Self-Similar Fractal Mosaics

Christoph Thäle

University of Fribourg, Department of Mathematics, Chemin du Musée 23, 1700 Fribourg/Switzerland

christoph.thaele@unifr.ch

Abstract

A new model for random tessellations having a fractal component is introduced. An explicit formula for the Hausdorff dimension is given and the exact gauge function of its Hausdorff measure is calculated. Moreover, fractal curvatures and mean fractal curvatures are considered. A theoretical result about the relation of these quantities is shown and is demonstrated numerically by an example..

1. The Model

A polyhedron P in $\mathbb{R}^d$ (which is not necessarily convex) is called self-similar, if there exists a natural number $n \in \mathbb{N}$ and polyhedra $P_1, \dots, P_n \subset \mathbb{R}^d$ similar to P, intersecting each other only in their boundaries, such that

$$P = \bigcup_{k=1}^{n} P_k.$$

If in addition all P_k are congruent, we call P replicating. Examples for replicating polyhedra are the cubes $[0,1]^d$ or the standard simplices Δ^d.

By a tessellation or a mosaic M of $\mathbb{R}^d$ we understand a countable family of polyhedra $P_1, P_2, \dots$ (often called cells) with following properties:

- $\bigcup_{k=1}^{\infty} P_k = \mathbb{R}^d$,
- a bounded set in $\mathbb{R}^d$ intersects only a finite number of polyhedra,
- the interiors of two different polyhedra are disjoint.

A tessellation is called self-similar, if all polyhedra are congruent (this assumption can be omitted, but is included for holding formulas simpler) and each polyhedron P_k is self-similar and replicating, if each P_k admits a replicating dissection as described above. An example for a self-similar tessellation is given by a tessellation, whose cells P_k are congruent (squares or higher dimensional) cubes. More complicated examples were constructed in [2].

Let M be a self-similar tessellation. To each cell P_{k_1} ($k_1 \in \mathbb{N}$) of M we associate a random variable X_{k_1} with

$$\mathbb{P}(X_{k_1} = 1) = 1 - \mathbb{P}(X_{k_1} = 0) = p,$$

where $p \in [0,1]$ is a fixed model parameter. A random tessellation $I_1(M,p)$ is now obtained from the deterministic mosaic M by dividing all cells P_{k_1} with $X_{k_1} = 1$ according to their fixed self-similar dissection. The cells of $I_1(M,p)$ are denoted by $P_{k_1k_2}$ and we associate with each $P_{k_1k_2}$ the random variable $X_{k_1k_2}$ with the same distribution as the X_{k_1} from above. The tessellation $I_2(M,p)$ is now obtained from $I_1(M,p)$ by dividing all cells $P_{k_1k_2}$ according to their fixed self-similar dissection iff $X_{k_1} \cdot X_{k_1k_2} = 1$, i.e. none of the factors equals zero. Proceeding in this way we obtain the $(n+1)$-fold random iteration $I_{n+1}(M,p)$ from $I_n(M,p)$ by dividing all cells $P_{k_1 \dots k_n}$ iff $X_{k_1} \cdots X_{k_n} \neq 0$. For $n = 1,2,3,4$ this method is illustrated as follows:

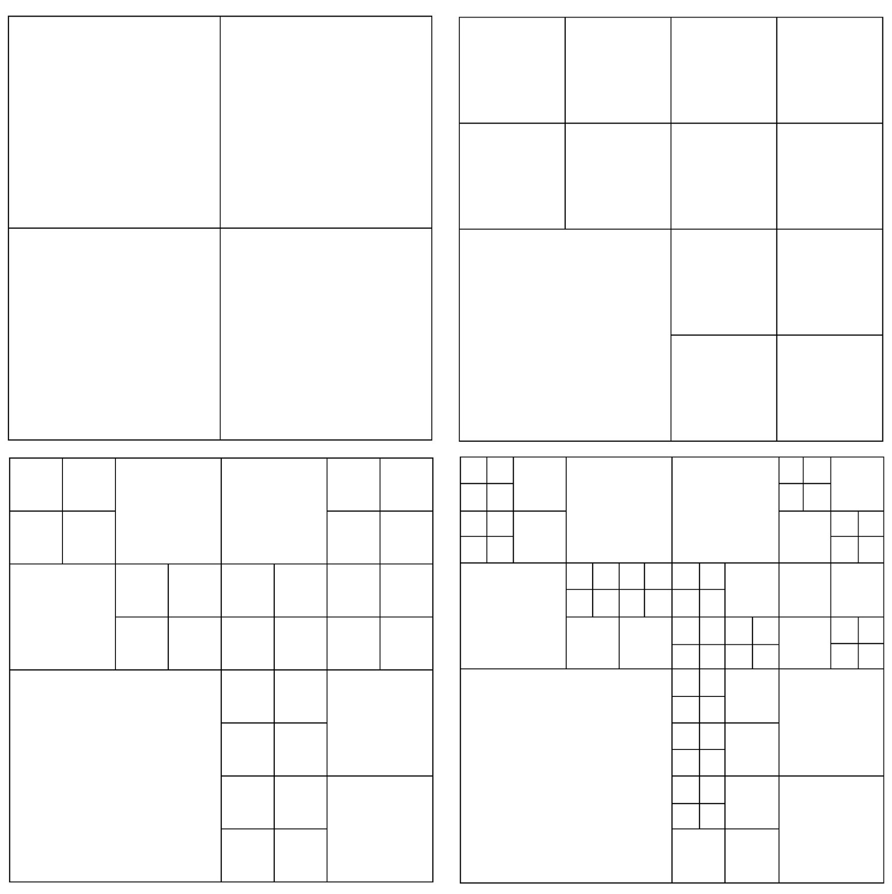

Figure 1: *The random iterations $I_n(M,p)$, $n = 1,2,3,4$ for a square tessellation with $p = 1/2$*

We are here interested in the limit as n tends to infinity. We therefore define the limit set

$$I_\infty(M,p) := \overline{\bigcup_{n=1}^{\infty} \{\partial P_{k_1 \dots k_n} : P_{k_1 \dots k_n} \in I_n(M,p)\}},$$

where ∂P is the boundary of the cell P and $\overline{A}$ denotes the topological closure of a set A. Some limit sets of a 9-replicating tessellation of $\mathbb{R}^2$ are illustrated below:

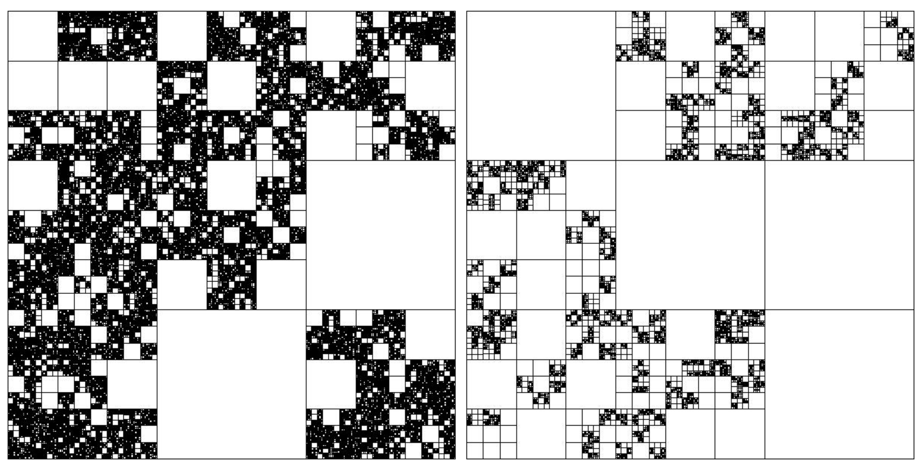

Figure 2: *A limit set for $p = 0.8$ and $p = 0.5$*

2. Hausdorff Dimension and Exact Gauge Function

Let $A \subset \mathbb{R}^d$ and $h : [0,\delta] \to [0,\infty)$, for some $\delta > 0$, be right continuous and non-decreasing with $h(t) > 0$ for $t > 0$ and $\lim_{t \to 0} h(t) = 0$. Assume furthermore that there exists a constant $K > 0$, such that $h(2t) \leq Kh(t)$. The Hausdorff h-measure $\mathcal{H}^h(A)$ of $A \subset \mathbb{R}^d$ wrt. the gauge function h is defined by

$$\mathcal{H}^h(A) := \lim_{\delta \to 0} \inf \left\{ \sum_{k=1}^{\infty} h(|A_k|) : A \subset \bigcup_{k=1}^{\infty} A_k, |A_k| < \delta \right\},$$

where the infimum is taken over all coverings $(A_k)_{k=1}^{\infty}$ of A. In the definition, $|A_k|$ is the diameter of the set A_k. A gauge function h is called exact for a subset $A \subset \mathbb{R}^d$, iff $0 < \mathcal{H}^h(A) < \infty$. The Hausdorff dimension $dim_H\, A$ of a set $A \subset \mathbb{R}^d$ is the unique $D \geq 0$ with the property that

$$\mathcal{H}^s(A) = \begin{cases} +\infty & : s < D \\ 0 & : s > D. \end{cases}$$

With a self-similar dissection $P = P_1 \cup \ldots \cup P_n$ of a polyhedron P there are associated n contraction ratios $r_1, \dots, r_n$, which are given by the relation $r_k P = P_k$, $k = 1, \dots, n$. In the case of a replicating dissection these contraction ratios are necessarily equal and we will write $r = r_1 = \ldots = r_n$ in this special case.

Theorem 1 *Let $p \in [0,1]$ and M be a self-similar tessellation of $\mathbb{R}^d$ with associated contraction ratios $r_1, \dots, r_n$ and assume that for all $k = 1, \dots, n$ we have $0 < r_k < 1$. Then $I_\infty(M,p)$ has Hausdorff dimension $D = \max\{d-1, \alpha\}$ and α obeys*

$$\sum_{k=1}^{n} r_k^{\alpha} = \frac{1}{p}.$$

Moreover,

$$h(t) = t^{\alpha} \left(\ln \ln \frac{1}{t} \right)^{1 - \frac{\alpha}{d}}$$

is a exact gauge function for $I_\infty(M,p)$, if $(\sum_{k=1}^{n} r_i)^{-1} \leq p$ and $h(t) = t^{d-1}$ is an exact gauge function in the case $(\sum_{k=1}^{n} r_i)^{-1} > p$.

In the special case of a replicating tessellation M of $\mathbb{R}^d$ we can give a more explicit formula for the Hausdorff dimension:

$$D = dim_H\, I_\infty(M,p) = \max\left\{ d-1, d - \frac{\ln p}{\ln r} \right\}.$$

For the exact gauge function we obtain

$$h(t) = \begin{cases} t^D \left(\ln \ln \frac{1}{t} \right)^{1 - \frac{D}{d}} & : p \geq r \\ t^{d-1} & : p < r. \end{cases}$$

As an example we regard again the 9-replicating tessellation of $\mathbb{R}^2$ by squares.

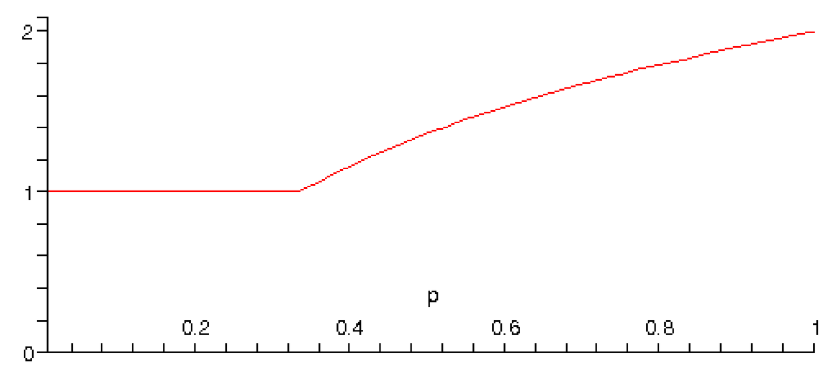

Figure 3: *Hausdorff dimension of the limit set as a function of $p \in [0,1]$*

3. Fractal Curvatures and Mean Fractal Curvatures

Let $A \subset \mathbb{R}^d$ be polyconvex and $r > 0$. Then the volume of the r-parallel set $A_r := \{x \in \mathbb{R}^d : dist(x,A) \leq r\}$ can be expressed as a polynomial of degree d:

$$vol(A_r) = \sum_{k=0}^{d} \omega_{d-k} C_k(A) r^{d-k},$$

where ω_k is the volume of the k-dimensional unit ball. The coefficients C_k are called curvatures (or intrinsic volumes) of the set A.

We define the fractal curvatures $C_k^f(I_\infty(M,p))$ for $k = 0, \dots, d$ of the limit set $I_\infty(M,p)$ of a d-dimensional self-similar mosaic M by

$$C_k^f(I_\infty(M,p)) := \lim_{\delta \to 0} \frac{1}{-\ln \delta} \int_{\delta}^{1} r^{D-k} C_k((I_\infty(M,p))_r) \frac{dr}{r}.$$

Note that the curvatures $C_k((I_\infty(M,p))_r)$ are well defined, since $(I_\infty(M,p))_r$ is polyconvex for any $r > 0$. This approach leads in $\mathbb{R}^d$ to $d+1$ parameters, which reflect the geometry of the fractal set $I_\infty(M,p)$. Since the limit sets $I_\infty(M,p)$ are random, the fractal curvatures $C_k^f(I_\infty(M,p))$ are random, too. But using the renewal theorem for branching random walks one can show that $C_k^f(I_\infty(M,p))$ is a random multiple, which only depends on the distribution of $I_\infty(M,p)$ but not on k, of a deterministic integral I. So we can write

$$C_k^f(I_\infty(M,p)(\omega)) = X(\omega) \cdot I(k,p,M).$$

This shows that the quotients

$$c_{k,l} := \frac{C_k^f(I_\infty(M,p))}{C_l^f(I_\infty(M,p))} = \frac{I(k,p,M)}{I(l,p,M)}$$

are deterministic constants and we are now going to characterize them in terms of the mean fractal curvatures of $I_\infty(M,p)$. They are defined by the same idea as above. For $k = 0, \dots, d$, we put

$$\overline{C_k^f}(I_\infty(M,p)) := \lim_{\delta \to 0} \frac{1}{-\ln \delta} \int_{\delta}^{1} r^{D-k} \mathbb{E} C_k((I_\infty(M,p))_r) \frac{dr}{r}.$$

Theorem 2 *For the limit set $I_\infty(M,p)$ of a random self-similar mosaic M in $\mathbb{R}^d$, the fractal curvatures and mean fractal curvatures $C_k^f(I_\infty(M,p))$ and $\overline{C_k^f}(I_\infty(M,p))$ exist and are positive and finite. Moreover*

$$\frac{C_k^f(I_\infty(M,p))}{C_l^f(I_\infty(M,p))} = \frac{\overline{C_k^f}(I_\infty(M,p))}{\overline{C_l^f}(I_\infty(M,p))} = \frac{I(k,p,M)}{I(l,p,M)} := c_{k,l}$$

is a constant only depending on k, l, M and p.

This shows, that from only one measuring we can obtain a mean value, i.e. the mean value over the full sample.

To demonstrate this numerically, we come back to our square tiling example from above. We estimate the Hausdorff dimension and the fractal curvatures C_0^f (fractal Euler number), C_1^f (fractal boundary length) and C_2^f (Minkowski content) for the parameter $p = \frac{1}{2}$ of 6 different realizations of $I_\infty(M,p)$. The results are summarized in the following table:

Nr.	dim_H	C_0^f	C_1^f	C_2^f
1	1.5090	−1893.78	18340.17	64169.08
2	1.5519	−2396.06	21096.40	77923.86
3	1.4452	−1690.27	17737.18	57018.47
4	1.5069	−1898.76	18393.58	63596.43
5	1.4164	−1221.33	14481.39	45134.82
6	1.4505	−1498.31	16437.67	53719.87

From these values we can now compute the quotients $c_{k,l}$:

Nr.	$c_{0,1}$	$c_{0,2}$	$c_{1,2}$
1	−0.095295	−0.029644	0.311077
2	−0.113576	−0.030748	0.270730
3	−0.103259	−0.029512	0.285810
4	−0.103229	−0.029856	0.289223
5	−0.084338	−0.027059	0.320847
6	−0.091151	−0.027891	0.305988

The values confirm experimentally the theoretical results from Theorem 2 and show that the numerical methods – which were originally developed for the deterministic case – seem to be stable with respect to random perturbations. Moreover, they seem to work even more accurate in the random case, which was already expected. The values were calculated unsing the GeoStoch software package developed at the University of Ulm.

References

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