Multifractal textures on the sphere: Computer graphics, Astrophysics and Analysis

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Multifractality (= self-similarity):

Romanesco cabbage

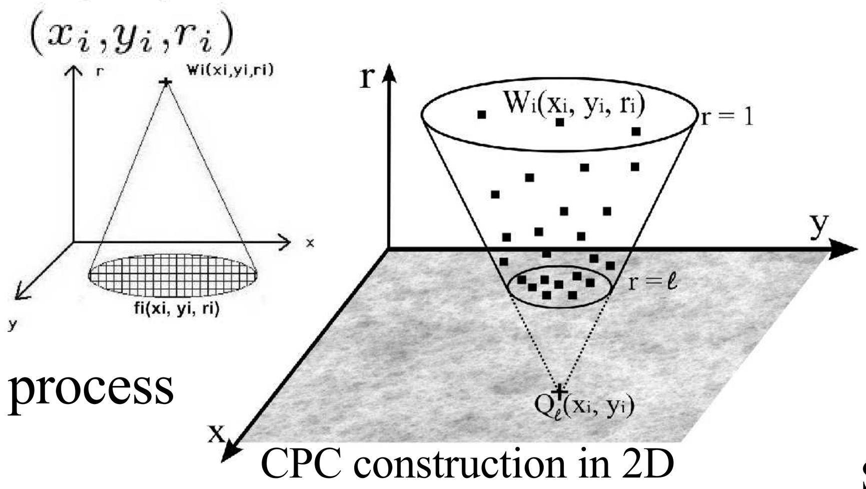
Sand

- **Deterministic:**
  - an object is identical to its deterministic transform (ex: using dilation or rotation)
- **Statistical:**
  - after transformation (ex: dilation), a part of the object is statistically identical to the whole
  - characterized by a set of multifractal exponents $\zeta(q)$
  - if $\zeta(q) = qH$ the object is called monofractal
  - if $\zeta(q) = qH + \tau(q)$ the object is called multifractal and $\tau(q)$ is a non-linear function
- **Multifractal objects are usually constructed using multiplicative cascades**

Multiplicative cascades on the sphere:

- **Compound Poisson Cascades in 2D:**

$$Q_\ell(x,y) = \alpha \prod_{(x_i,y_i,r_i)} W_i^{f_i(x,y)} \propto \exp \sum_{(x_i,y_i,r_i)} \ln W_i f_i(x,y)$$

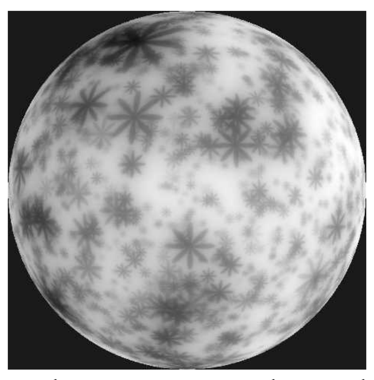


CPC construction in 2D

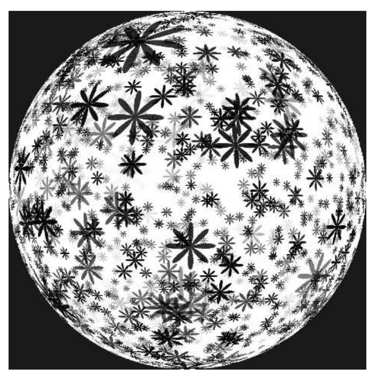
- (x_i, y_i, r_i) result from a Poisson point process
- wheighted by multipliers $W_i > 0$
- α is a normalisation coefficient
- $f_i(x,y)$ is a pattern function
- $r_i\,(\ell \le r_i \le 1)$ reads as a scale parameter
- distributed with density $1/r_i^3$

- creates purely multifractal objects
- with $\tau(q) = q(\mathbb{E}W_i - 1) + 1 - \mathbb{E}W_i^q$
- use of a pseudo-fractionnal integration
  - low-pass filtering in $1/\|k\|^H$
  - then characterized by $\zeta(q) = qH + \tau(q)$

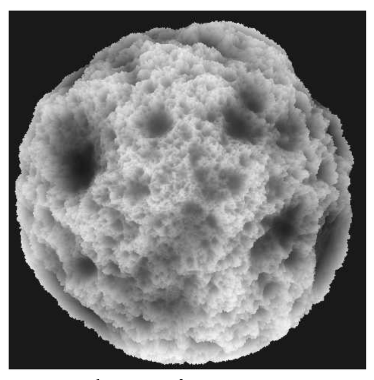
- **Various degrees of freedom:** W_i distribution, pattern, characteristic scale, anisotropy or inhomogeneity...



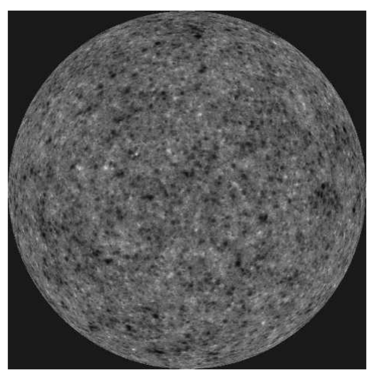
Filtered texture on the sphere



CPC texture on the sphere

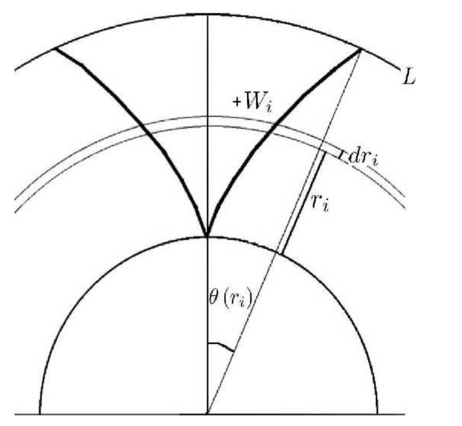


Elevation map



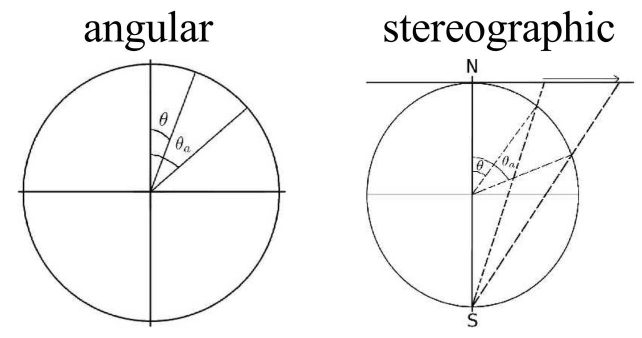
Synthetic solar texture

- **Sphere: the simplest curved surface**

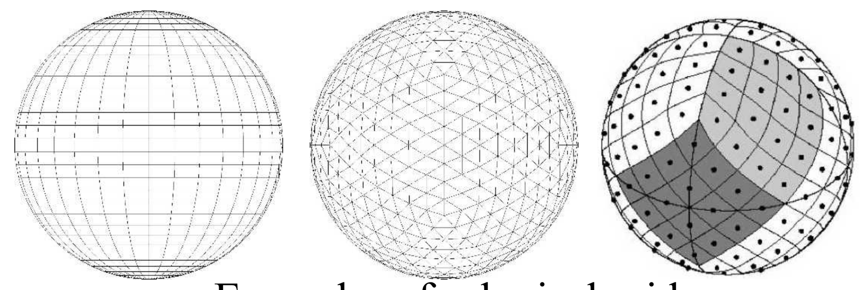


Spherical CPC construction

- synthesis directly on the sphere
- no poles or seam artifacts
- spherical dilation non unique



- pseudo-fractionnal integration
  - filtering in the spherical harmonics domain
  - need to define a spherical grid

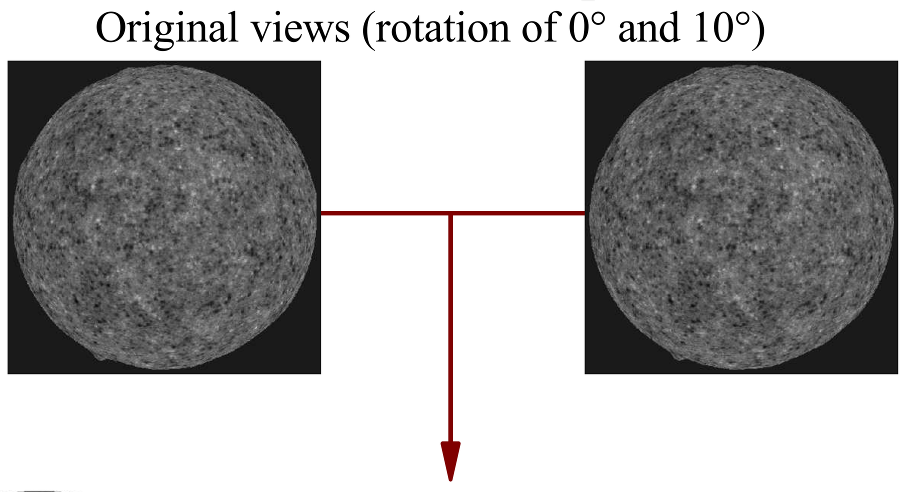


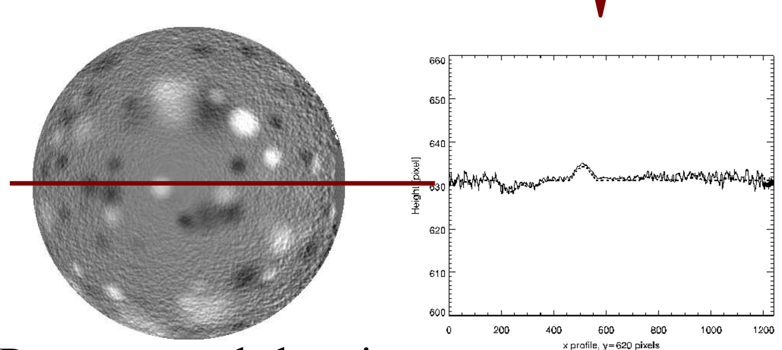
Examples of spherical grids

Stereographic reconstruction of the Sun:

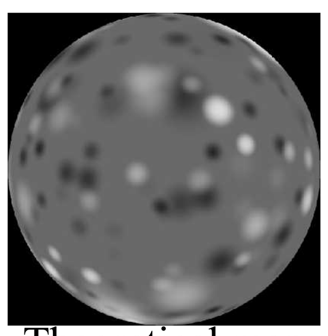
- **Validation of a stereographic image processing algorithm**
  - 2 images of the Quiet Sun from the STEREO mission of the ESA/NASA
  - estimation of the apparent radius (altitude)
- **Test bench:**
  - spheres with a synthetic solar texture (CPC texture)
  - use of a known elevation map

Original views (rotation of 0° and 10°)





Reconstructed elevation map and its x-profile

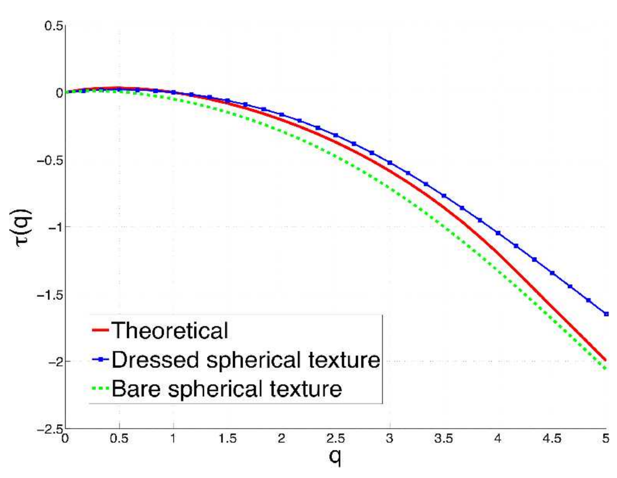


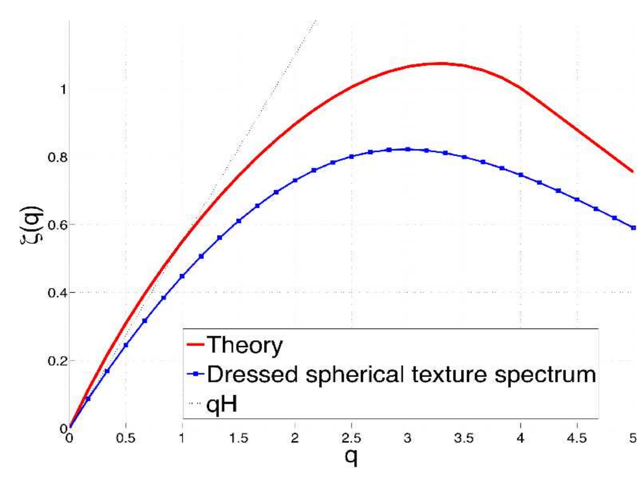
Theoretical map

Multifractal analysis using spherical wavelets:

$$S_q^{swt}(a_j) = \frac{1}{N_j} \sum |T_X(j)|^q \propto a_j^{\zeta(q)}$$

- $S_q^{swt}(a_j)$ is a partition function computed from the spherical wavelet coefficients $T_X(j)$ of the function X for a moment order q and a scale a_j
- construction of spherical wavelets ⇨ choice of the spherical grid





Estimation of $\tau(q)$ and $\zeta(q)$ using $S_q^{swt}(a)$

- **Results have a good trend but also a systematic bias**

References :

[1] P. Chainais, « Infinitely divisible cascades to model the statistics of natural images », IEEE PAMI, 2007

[2] E. Koenig et al., « Synthèse de textures sur la sphère », MajecSTIC'08, 2008

[3] S. Gissot et al., « 3D reconstruction from SECCHI-EUVI images using an optical-flow algorithm », Solar Physics, 2008

[4] E. Koenig et al., « Multifractal analysis on the sphere », ICISP'08, 2008

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