Multifractional Stochastic Volatility Models

Antoine Ayache and Qidi Peng

Laboratory Paul Painlevé (USTL),
Antoine.Ayache@math.univ-lille1.fr, Qidi.Peng@math.univ-lille1.fr

15/01/2009

1. Introduction and motivation

2. Main ideas and principal results

- Stochastic Volatility Models (SVM) are extensions of the well known Black and Scholes model,

$$\mathrm{d}S(t) = \mu S(t) + \sigma S(t)\,\mathrm{d}W(t),$$

  in which the volatility (constant) σ is replaced by a stochastic process.

- Hull and White (1987) and other authors in mathematical finance have introduced them in order to account the volatility effects of exogenous arrivals of information.

$\rightarrowtail$ Gloter and Hoffmann are interested in statistical inference in a parametric stochastic volatility model driven by a fractional Brownian motion (fSVM). Namely, the model considered can be expressed as :

$$Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s) \qquad (1)$$

where

- $\{Z(t)\}_{t\in[0,1]}$ denotes the logarithm of the price of the underlying asset (z_0 is determinist).
- $\{W(s)\}_{s\in[0,1]}$ is a standard Brownian motion (Bm).
- $\{\sigma(s)\}_{s\in[0,1]}$ denotes volatility process.

We assume that $\{\sigma(s)\}_{s\in[0,1]}$ is independent of $\{W(s)\}_{s\in[0,1]}$ and it is expressed as :

$$\sigma(s) = \sigma_0 + \Phi(\theta, B_\alpha(s)), \tag{2}$$

where

- θ is an unknown parameter in Θ, a fixed compact interval of $\mathbb{R}$.
- σ_0 is a real number and $\Phi : \Theta \times \mathbb{R} \to \mathbb{R}$ is a function : for all $x \in \mathbb{R}$, we set

$$f(x) := \Big(\sigma_0 + \Phi(\theta, x)\Big)^2. \tag{3}$$

From now on, we suppose that $f \in C^2_{pol}(K')$ where K' does not depend on θ. For each integer $l \geq 0$ and each real $K > 0$,

$$C^l_{pol}(K) := \Big\{ h \in C^l(\mathbb{R}) : \text{for all} \quad x \in \mathbb{R}, \\ |h(x)| + \cdots + |h^{(l)}(x)| \leq K(1 + |x|^K) \Big\}. \tag{4}$$

- $\{B_\alpha(s)\}_{s\in[0,1]}$ is the fractional Brownian motion (fBm) with Hurst parameter $\alpha \in]0,1[$.

The fBm $\{B_\alpha(s)\}_{s\in[0,1]}$ is the continuous centred Gaussian process whose covariance is given as, for all $s,t \in [0,1]$,

$$\mathbb{E}\left(B_\alpha(s)B_\alpha(t)\right) = \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}).$$

$\rightarrowtail$ When $\alpha > 1/2$, B_α is a functional primitive of Bm, when $\alpha < 1/2$, B_α is a functional derivative of Bm and $B_{1/2}$ is the Bm.

$\rightarrowtail$ When $\alpha \neq 1/2$ the increments of B_α are correlated, it has the long memory property when $\alpha > 1/2$.

That's the reason why Comte and Renault have proposed to introduce the fBm in the MSV.

- Under some additional technique assumptions, Gloter and Hoffmann have constructed optimal contrast estimators of the unknown parameter θ starting from the high frequency data $Z(j/n)$, $j = 0, \ldots, n$.

- A very important step in the method they used, was the construction of estimators of integrated functionals of the volatility of the form :

$$\int_0^1 f'(B_\alpha(s))h(\sigma^2(s))\,\mathrm{d}s, \tag{5}$$

where $h \in C^1_{pol}(K)$.

- The problem of their estimation is also of some importance in its own right because more or less similar integrals appear in some options pricing formulas.

Multifractional Stochasitic Volatility models(MSVM) :

$$\begin{cases} Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s) \\ \sigma(s) = \sigma_0 + \Phi(\theta, X(s)), \end{cases} \tag{6}$$

where $\{X(s)\}_{s\in[0,1]}$ in a multifractional Brownian motion (mBm).

The MSVM generalize fSVM.

In order to explain the interests of these new models, make some recalls for the mBm.

- Harmonizable representation of fBm : The mBm has been introduced independently in one hand by Peltier and Lévy Véhel (1995) and in the other hand by Benassi, Jaffard and Roux (1997).

- The mBm $\{X(s)\}_{s\in[0,1]}$ is defined as

$$B_\alpha(s) := \int_{\mathbb{R}} \frac{e^{is\xi}-1}{|\xi|^{\alpha+1/2}}\,\mathrm{d}\hat{B}(\xi), \tag{7}$$

where $\mathrm{d}\hat{B}$ denote the "transformation of Fourier" of the white noise $\mathrm{d}B$: for all $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(s)\,\mathrm{d}B(s) = \int_{\mathbb{R}} \hat{f}(\xi)\,\mathrm{d}\hat{B}(\xi).$$

- $X = \{X(s)\}_{s\in[0,1]}$ the mBm (Peltier, Lévy Véhel and Benassi, Jaffard, Roux) is defined as :

$$X(s) := \int_{\mathbb{R}} \frac{e^{is\xi} - 1}{|\xi|^{H(s)+1/2}}\, \mathrm{d}\hat{B}(\xi), \tag{8}$$

  where $s \mapsto H(s)$ is a continuous function with values in $]0, 1[$. $H(.)$ should be continuous in order that the Gaussian process X is of continuous trajectories.

- We assume also that $H(.)$ is a C^2-function in order to estimate the correlations between the generalized increments of local averages of mBm.

$\rightarrowtail$ The mBm is an extension of fBm :

- When $H(.)$ is constant we get the fBm.
- For all $s_0 \in [0,1]$, when s is in the neighborhood of s_0, $X(s) \backsimeq B_{H(s_0)}(s)$.
- Contrarily to fBm, the local Hölder regularities of mBm are allowed to change with time.
  the local Hölder regularity of a process $Y = \{Y(s)\}_{s\in[0,1]}$ in neighborhood of a point s_0 is measured by $h_Y(s_0)$ the pointwise Hölder exponent (PHE) of Y on s_0.

When Y is a continuous process and nowhere derivable $h_Y(s_0)$ is defined by :

$$h_Y(s_0) = \sup\left\{h \geq 0 : \limsup_{\mu \to 0} \frac{|X(s_0+\mu) - X(s_0)|}{|\mu|^h} = 0\right\}.$$

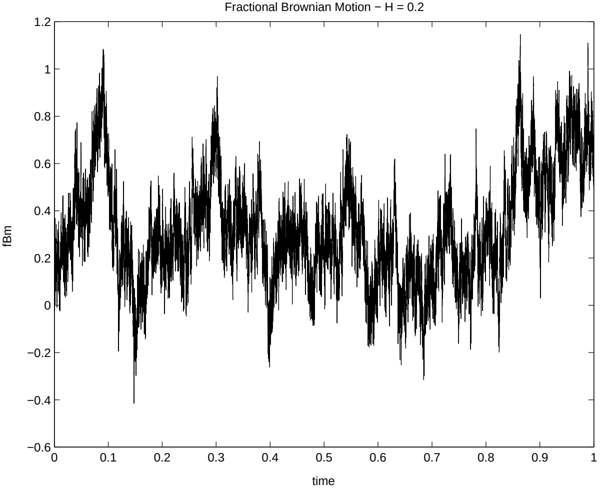
We have $0 \leq h_Y(s_0) \leq 1$ and the closer $h_Y(s_0)$ is to 1, the more regular the processus Y is in the neighborhood of s_0.

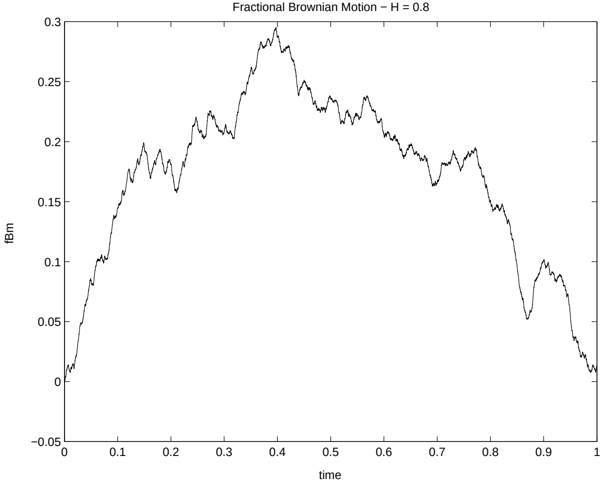
- h_X The PHE of mBm X verifies a.s. for all s_0

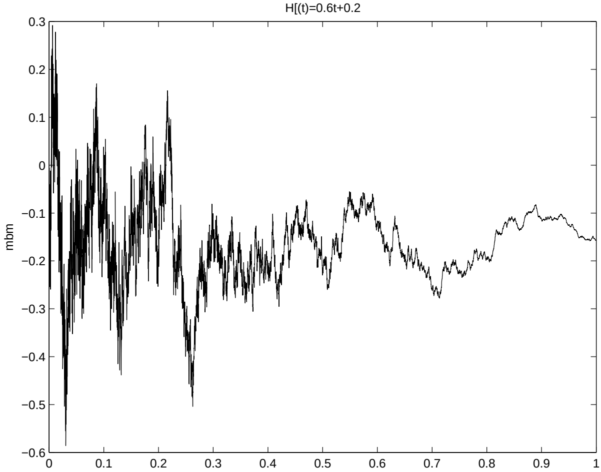
$$h_X(s_0) = H(s_0). \tag{9}$$

Simulation

trajectories of fBm and of mBm :




Fractional Brownian Motion − H = 0.8
fBm
time
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


H[(t)=0.6t+0.2
mbm
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$\rightarrowtail$ To obtain our main results, basically, we will use some technics which are more or less similar to the work of Gloter and Hoffmann.

$\rightarrowtail$ However new difficulties appear in our multifractional new setting. These new difficulties are essentially due to the fact that the dependence structure of mBm is much more complicated than that of fBm :

- the fBm is globally self-similar : for all real $a > 0$:

$$\{B_\alpha(as)\}_{s\in\mathbb{R}} \overset{law}{=} \{a^\alpha B_\alpha(s)\}_{s\in\mathbb{R}},$$

but the mBm is only locally asymptotically auto-similar : for all $s_0 \in \mathbb{R}$,

$$\lim_{p\to 0_+} \mathbf{law}\Big\{\frac{X(s_0+p\mu)-X(s_0)}{p^{H(s_0)}}\Big\}_{\mu\in\mathbb{R}} = \mathbf{law}\{B_{H(s_0)}(\mu)\}_{\mu\in\mathbb{R}}. \tag{10}$$

On each s_0, a fBm with Hurst parameter $H(s_0)$ "is tangent" to the mBm X.

- contrarily to fBm, the increments of mBm are not stationary : that's why its local Hölder regularity can change with time.

- The covariance of mBm is much more complicated than that of fBm : for all $s_1, s_2 \in \mathbb{R}$,

$$\mathbb{E}\left(X(s_1)X(s_2)\right) = A(H(s_1) + H(s_2)) \times \\ \times \left\{|s_1|^{H(s_1)+H(s_2)} + |s_2|^{H(s_1)+H(s_2)} - |s_1 - s_2|^{H(s_1)+H(s_2)}\right\}, \tag{11}$$

where $A(.)$ denote a regular function and strictly positive.

The estimation of correlations between the generalized increments of local averages of mBm will demand much more work than in the case of fBm.

We are interested in statistical inference in a stochastic volatility model driven by a multifractional Brownian motion (mBm).
Our main objectif : in the linear case i.e. $\Phi(\theta, x) = \theta x$, construct an estimator of the unknown parameter θ starting from the observation of $Z(j/n), j = 0, \cdots, n$.
Important Step($\mathscr{E}$) : Construct an estimator of the integrated functionals of the volatility :

$$V(h) := \int_0^1 f'(X(s))^2 h(Y(s))\,\mathrm{d}s, \tag{12}$$

where

$$Y(s) := f(X(s)) := \Big(\sigma_0 + \Phi(\theta, X(s))\Big)^2,$$

and $h \in C^1_{pol}(K)$. This step is also of some importance in its own right (in some options pricing formulas).

Construction of an estimator of $V(h)$ starting from $Z(j/n), j = 0, \cdots, n$:
The main ideas :

- An important difficulty is due to the fact that $V(h)$ depends on the hidden process $\{Y(s)\} = \{\sigma^2(s)\}$.

- The $Z(j/n), j = 0, \cdots, n$ allows to estimate the $\overline{Y}_{i,N_n}, i = 0, \cdots, N_n - 1$, where $\overline{Y}_{i,N_n}$ is the average value of Y in the interval $[\frac{i}{N_n}, \frac{i+1}{N_n}]$:

$$\overline{Y}_{i,N_n} := N_n \int_{i/N_n}^{(i+1)/N_n} Y(s)\,\mathrm{d}s. \qquad (13)$$

$N_n < n$. The choice of N_n will be defined more precisely later.

For all $i = 0, \cdots, N_n$ set $j_i := [in/N_n]$ and when $i \leq N_n - 1$ set

$$\hat{Y}_{i,N_n,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \Big(Z((j_i+k+1)/n) - Z((j_i+k)/n) \Big)^2. \quad (14)$$

Given $Z(t) = z_0 + \int_0^t \sigma(s)\,\mathrm{d}W(s)$, it follows from the Itô formula that

$$\hat{Y}_{i,N_n,n} = \overline{Y}_{i,N_n} + R_{i,N_n}, \quad (15)$$

where

$$\begin{aligned} R_{i,N_n} := \quad & 2N_n \sum_{k=0}^{j_{i+1}-j_i-1} \int_{(j_i+k)/n}^{(j_i+k+1)/n} (Z(s) - Z(\frac{j_i+k}{n}))\sigma(s)\,\mathrm{d}W(s) \\ & +2N_n \int_{j_i/n}^{i/N_n} Y(s)\,\mathrm{d}s - 2N_n \int_{j_{i+1}}^{(i+1)/N_n} Y(s)\,\mathrm{d}s, \end{aligned}$$

is the rest term which we can neglect by a good choice of N_n.

Given an estimator of $V(h)$ starting from $\overline{Y}_{i,N}$.

- $a = (a_0, \ldots, a_p)$ is an arbitrary fixed sequence of $p+1$ reals whose first $M(a)$ moments are zero : for all $l \in \{0, \cdots, M(a)-1\}$,

$$\sum_{k=0}^{p} k^l a_k = 0, \text{for all } l = 0, \ldots, M(a) - 1.$$

- The generalized increments (corresponding to a) of the sequence $(\overline{Y}_{i,N})_{0 \leq i \leq N}$ are defined for all $i \in \{0, \cdots, N-p-1\}$ as

$$\Delta_a \overline{Y}_{i,N} := \sum_{k=0}^{p} a_k \overline{Y}_{i+k,N}.$$

While $a = (1, -1)$ we get the increments of order 1 and while $a = (1, -2, 1)$ we get that of order 2.

$\rightarrowtail$ $\Delta_a \hat{Y}_{i,N_n,n}, i = 0, \cdots, N_n - p - 1$ the generalized increments of the sequence $(\hat{Y}_{i,N_n,n})_{0 \le i \le N_n}$ are defined with the same manner. Similarly for $\Delta_a \overline{X}_{i,N}, i = 0, \cdots, N - p - 1$ the generalized increments of the sequence $(\overline{X}_{i,N})_{0 \le i \le N}$.

Replace the increments of order 1 by the generalized increments is an important idea which has already been introduced by Istas and Lang (97) in the domain of the estimation of Hölder exponents of the Gaussian process. Usually the bigger $M(a)$ is the less stronger the correlations between the generalized increments.

Theorem 1

For all integer $N \geq p+1$ and all $h \in C^1_{pol}(K)$, let

$$V_N(h) := N^{-1} \sum_{i=0}^{N-p-1} \frac{(\Delta_a \overline{Y}_{i,N})^2}{Var(\Delta_a \overline{X}_{i,N})} h(\overline{Y}_{i,N}), \qquad (16)$$

then there exists a constant $c > 0$ such that for all $N \geq p+1$,

$$\mathbb{E}\{|V_N(h) - V(h)|\} \leq cN^{-H_*}, \qquad (17)$$

where $H_* = \min\{H(s) : s \in [0,1]\}$.

Remark

H_* is in fact the global Hölder exponent of the mBm in $[0,1]$; (17) gives a thought that the speed of convergence of $V_N(h)$ is determined by the global Hölder regularity of the mBm in $[0,1]$.

To get an estimator of $V(h)$ starting from $Z(j/n), j = 0, \cdots, n$, a natural idea consists to replace in $V_{N_n}(h)$ the $\overline{Y}_{i,N_n}$ by $\hat{Y}_{i,N_n,n}$. But

$$N_n^{-1} \sum_{i=0}^{N-p-1} \frac{(\Delta_a \hat{Y}_{i,N_n,n})^2}{\mathrm{Var}(\Delta_a \overline{X}_{i,N_n})} h(\hat{Y}_{i,N_n,n}), \tag{18}$$

does not converge to $V(h)$. However, by adding to (18), the correction term

$$N_n^{-1} \sum_{i=0}^{N_n-p-1} -\frac{2\|a\|_2^2}{\mathrm{Var}(\Delta_a \overline{X}_{i,N_n}) m_n} (\hat{Y}_{i,N_n,n})^2 h(\hat{Y}_{i,N_n,n}), \tag{19}$$

where $m_n = [n/N_n]$ and $\|a\|_2 := \left(\sum_{k=0}^{p} |a_k|^2 \right)^{1/2}$. We can, in some condition, obtain an estimator of $V(h)$, which we note as $\hat{V}_n(h)$.

τ_1 and τ_2 denote two arbitrary and fixed real numbers verifying : for all $s \in [0,1]$,

$$0 < \tau_1 \leq H(s) \leq \tau_2 < 1. \tag{20}$$

Let $\delta' := \min(\tau_1, 1-\tau_1)$ and $\delta'' := \max(0, \tau_1 - 1/2)$. Recall that $m_n := [n/N_n]$.

Theorem 2

Assume that N_n is chosen such that

$$m_n^{-2\delta'} N_n^{2\tau_2+\delta''}, \tag{21}$$

remains bounded when $n \to +\infty$. We have then for n big enough,

$$\mathbb{E}\left\{|\hat{V}_n(h) - V(h)|\right\} = \mathcal{O}(N_n^{-\tau_1}). \tag{22}$$

Remark

When $X = B_\alpha$ with $\alpha \in]1/2, 1[$ and we take $\tau_1 = 1/2$ and $\tau_2 = \alpha$ we get a result of Gloter and Hoffmann.

Estimator starting from $Z(j/n), j = 0, \cdots, n$ of parameter θ of a linear model of MSV.
A such model is of the form

$$Z(t) = z_0 + \sigma_0 W(t) + \theta \int_0^t X(s)\,\mathrm{d}W(s), \tag{23}$$

where the mBm X is independent of Bm W.
For all integer n big enough, let

$$\hat{\theta}_n^2 := \frac{\hat{V}_n(1)}{4N_n^{-1} \sum_{i=0}^{N_n-p-1} \hat{Y}_{i,N_n,n}}. \tag{24}$$

Theorem 3

Assume that there exist two constants $0 < c_1 \leq c_2$ *such that for all* n *big enough,*

$$c_1 n^{2\delta'/(2\tau_2+2\delta'+\delta'')} \leq N_n \leq c_2 n^{2\delta'/(2\tau_2+2\delta'+\delta'')}. \qquad (25)$$

Then the sequence of random variables

$$\Big(n^{2\tau_1\delta'/(2\tau_2+2\delta'+\delta'')} (\hat{\theta}_n^2 - \theta^2) \Big)_n$$

is bounded in probability i.e. we have

$$\lim_{\lambda \to +\infty} \limsup_{n \to +\infty} \mathbb{P}\Big\{ n^{2\tau_1\delta'/(2\tau_2+2\delta'+\delta'')} |\hat{\theta}_n^2 - \theta^2| > \lambda \Big\} = 0. \qquad (26)$$

Corollary 4

Assume that the parameter H(.) of mBm is with value in [1/2,1[and let $H^* = \max\{H(s) : s \in [0,1]\}$.
Suppose that there exist two constants $0 < c_1 \leq c_2$ *such that for all n big enough,*

$$c_1 n^{1/(2H^*+1)} \leq N_n \leq c_2 n^{1/(2H^*+1)}. \tag{27}$$

Then the sequence of random variables

$$\Big(n^{1/(4H^*+2)}(\hat{\theta}_n^2 - \theta^2)\Big)_n$$

is bounded in probability.

This corollary is a straightforward consequence of Theorem 3, it suffices to take $\tau_1 = 1/2$ and $\tau_2 = H^*$.

Remark

Corrolary 4 has been already obtained by Gloter and Hoffmann in the case when $X = B_\alpha$ with $\alpha \in]1/2, 1[$. In this case these two authors ont have also proved that $\hat{\theta}_n^2$ "converges" exactly with the speed $n^{-1/(4\alpha+2)}$ and this speed of convergence is optimal.

Conjecture 5

The speed $n^{-1/(4H^*+2)}$ is optimal for the estimation of parameter θ of a stochastic volatility model driven by a mBm with the parameter $H(.)$ which is valued in [1/2,1[.

The Conjecture 5 means that the optimal speed for the estimation of the parameted θ is not determined by the global Hölder regularity of the mBm in the interval $[0, 1]$, but uniquely by its highest local Hölder regularity in this interval. This phenomenon seems unusual.$\square$