Anisotropy and Radon Transforms for Gaussian fields with stationarity properties

Aline Bonami, Université d'Orléans

Anisotropic Gaussian models (stationary increments)

Gaussian field $X = \{X(t); t \in \mathbb{R}^d\}$ with zero mean and variogram

$$v(t) := \mathbb{E}((X(t) - X(0))^2) = \int_{\mathbb{R}^d} \sin^2(t.\xi/2) f(\xi)\, d\xi \ , \ \forall t \in \mathbb{R}^d$$

with f spectral density, that is even, positive.

$$X \textbf{ isotropic} \quad \Leftrightarrow \quad f \textbf{ radial}$$

Typical Example for Isotropy : the Fractional Brownian Motion

$$f(\xi) = \frac{c_H}{|\xi|^{2H+d}} \qquad\qquad v(t) = a_H |t|^{2H}.$$

Also self-similar.

Two typical examples for Anistropy

- $f(\xi) = |\xi|^{-2h-d} \qquad \text{with } h \text{ homogenous of degree } 0$
B.-Estrade 2003, Ayache-B.-Estrade 2005, Biermé-Richard 2008
- If self similar model,

$$f(\xi) = \frac{\Omega(\xi)}{|\xi|^{2h+d}} \qquad \text{with } \Omega \text{ homogenous of degree } 0$$

Istas 2007.

$$X(t) - X(0) = \int_{\mathbb{R}^d} (e^{it.\xi} - 1) f^{1/2}(\xi) \, dW(\xi).$$

Questions

The observation is the data $X(\mathbf{k}/N)$, with $\mathbf{k} = (k_1, k_2, \cdots, k_d)$, $0 \leq k_j \leq N$.

- Propose a test of anisotropy.
- Identify the unknown function.

Same questions for other models :
Chan and Wood 2000 : stationary Gaussian process with covariance function (H fixed)

$$\gamma(t) = 1 - \omega(t)|t|^{2H} + \text{Rest}.$$

Chatterjee and Anderes 2009 : $X(\varphi(t))$, with X fixed isotropic as above, φ a smooth change of variables (in Dimension 2).

The self-similar model (H fixed)

Observation : v is also self-similar

$$v(t) = \int \sin^2(t \cdot \xi/2) f(\xi) d\xi = \omega(t)|t|^{2H},$$

with ω homogeneous of degree 0. Moreover,

$$\omega(t) = c_d \int_{|\xi|=1} |t.\xi|^{2H} \Omega(\xi) d\xi.$$

We can identify ω instead of Ω.
Use deconvolution or inverse Fourier transform to get Ω.

Remarks :

- ω varies less than Ω.
- Dependance between different lines.

The self-similar model : Radon Transform

This is work in progress with José León.
From now on, the dimension is 2. We fix an even "window" in one variable ρ.

Let $\mathbf{e}_\phi = (\cos\phi, \sin\phi)$ be a unit vector. The windowed Radon transform of the random field X is the process

$$\mathcal{R}X(t,\phi) = \int_{\phi^\perp} X(s + t e_\phi)\rho(s)ds.$$

Proposition. $\mathbb{E}(\mathcal{R}X(t,\phi))^2 = c_\rho \Omega(\mathbf{e}_\phi)|t|^{2H+1} + \text{Rest}.$

Moreover, the processes $\mathcal{R}X(t,\phi_1)$ and $\mathcal{R}X(t,\phi_2)$ are asymptotically independent when directions are different.
Allows to propose estimators of the function Ω, with explicit asymptotic behavior.

A Central Limit Theorem

We consider the second increments $\Delta_2 X_\phi(k/N)$ of the processes $\mathcal{R}X(t,\phi)$.
Then $V_\phi(N) = \frac{1}{N}\sum_{k=0}^{N-2} |\Delta_2 X_\phi(k/N)|^2$.

Theorem [A. B., J. León].

- The expectation of $N^{2H+1} V_\phi(N)$ tends to $c\Omega(\mathbf{e}_\phi)$.
- Moreover, $N^{2H+1/2}(V_\phi(N) - \mathbb{E}(V_\phi(N))$ converges in law to $\mathcal{N}(0,\sigma^2)$.

The same is valid for the finite distributions of $V_\phi(N)$, as a process on the unit circle.
Possible generalizations to p-variations.
Assumptions weakened to asymptotic behavior of the spectral density

$$f(\xi) = \frac{\Omega(\xi)}{|\xi|^{2H+d}}(1 + o(1))$$

when $|\xi| \to \infty$.

Identification and Test of Anisotropy

Difficulties.

- Replace integrals by discrete sums.
- Large data available only on a small number of directions (horizontal, vertical, diagonal, etc.).

Same problem for $f(\xi) = |\xi|^{-2h-2}$ (in Dimension 2), with h homogeneous of degree 0.
Call H_{hor} and H_{vert} the two values of h in the horizontal and vertical directions.
Preprint of Biermé and Richard, which proposes a test of anisotropy, based on a test of $H_{\text{hor}} = H_{\text{vert}}$.
Exact computation of the asymptotic confidence level for $H_{\text{hor}} - H_{\text{vert}}$, based on the asymptotic independence between Radon transforms in the horizontal and vertical directions.