Scaling Properties of the Spread Harmonic Measures

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Outline of the talk

- Harmonic measure: accessibility
- Spread harmonic measures: transfer
- Irregular shapes: scaling properties
- Conclusion and perspectives

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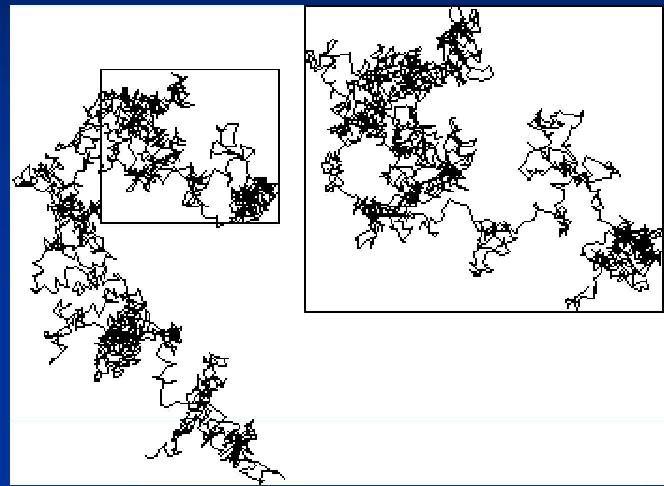
That amazing diffusion…

Random walk of a "blind" particle



Main transport mechanism:

- biology and physiology
- chemistry (heterogeneous catalysis)
- building industry (cement, concrete)

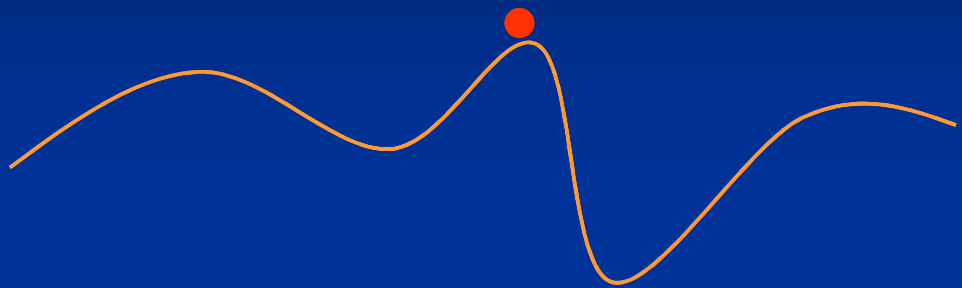
Harmonic measure

What is the accessibility of the boundary by Brownian motion?

$\omega_x\{A\}$ is the probability for Brownian motion started from point x to reach the boundary $\partial\Omega$ on the subset A for the first time

$T = \inf\{t > 0 : X_t \in \partial\Omega\}$ $\qquad$ $\omega_x\{A\} = P\{X_T \in A\}$

Partially reactive surface



probability σ for reaction, transfer, relaxation…

- Fi
- Fi
- Fi
- Finite resistivity of an electrode

What is the distribution of the transfer points?

$$\Lambda = D/W$$

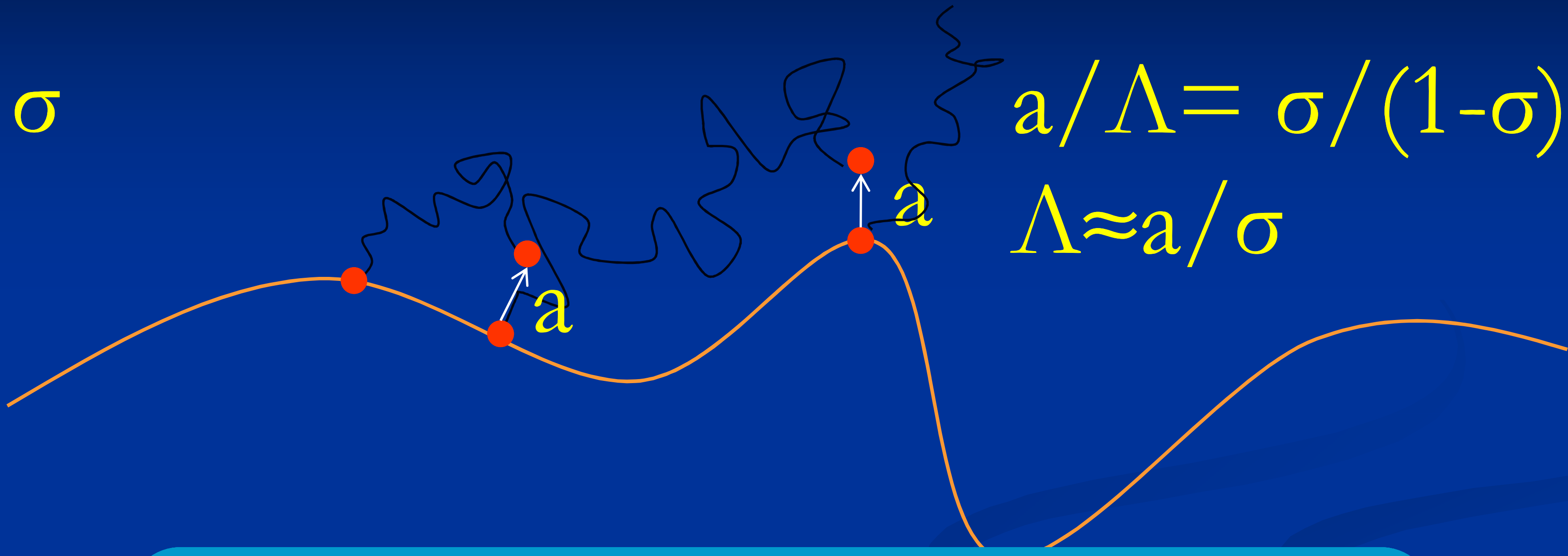
Sapoval, Phys. Rev. Lett. 73, 3314 (1994)

Partially reflected diffusion

σ

$a/\Lambda = \sigma/(1-\sigma)$

$\Lambda \approx a/\sigma$



What happens in the limit
$a \to 0,\ \sigma \to 0$?

Filoche, Sapoval, Eur. Phys. J. B 9, 755 (1999)
Grebenkov, Filoche, Sapoval, Eur. Phys. J. B 36, 221 (2003)

Reflected Brownian motion

Skorokhod equation

indicator function

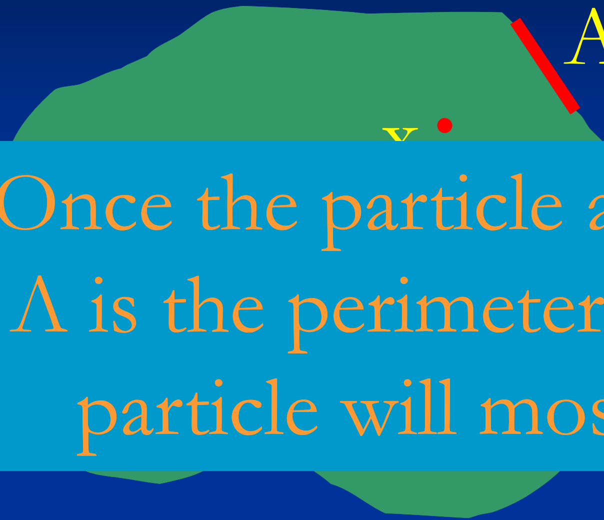
normal vector

$$dX_t = dW_t + I_{\partial\Omega}(X_t)\, n(X_t)\, d\ell_t$$

Brownian motion

local time on the boundary

Spread harmonic measures



Transfer across

Once the particle arrived onto the boundary, Λ is the perimeter of a region on which the particle will most probably be absorbed

$\omega_{x,\Lambda}\{A\}$ is the probability for partially reflected Brownian motion started from point x to be killed on the subset A of the boundary $\partial\Omega$

$T = \inf\{t>0 : l_t > \zeta\}$

$\omega_{x,\Lambda}\{A\} = P\{X_T \in A\}$

$P\{\zeta < x\} = \exp[-x/\Lambda]$

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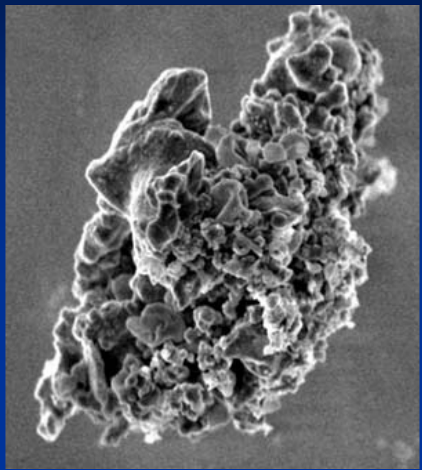
This was for smooth boundaries…

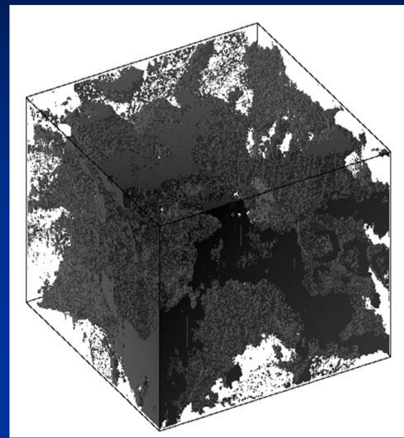
But what about diffusion in irregularly shaped domains?

Irregular shapes…

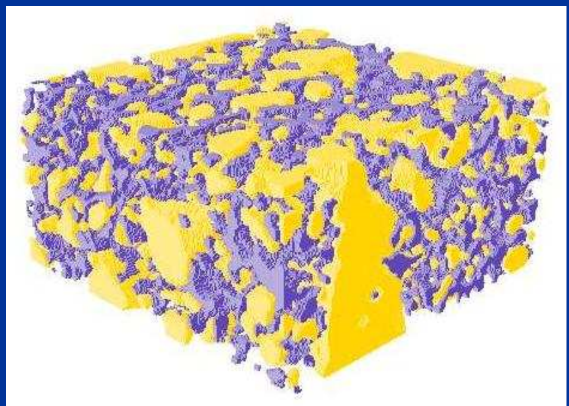
in material sciences



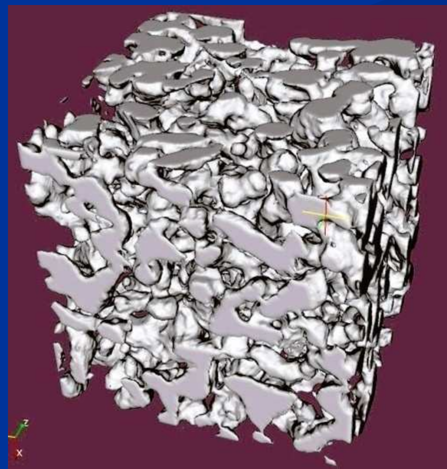
Stardust



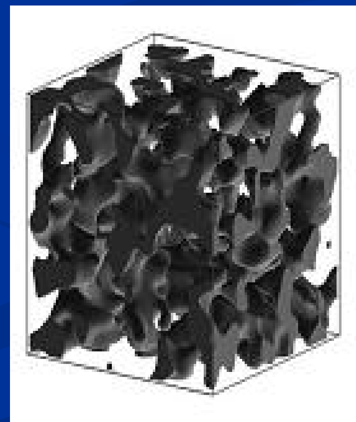
3D reconstruction of a limestone



3D reconstruction of a cement paste



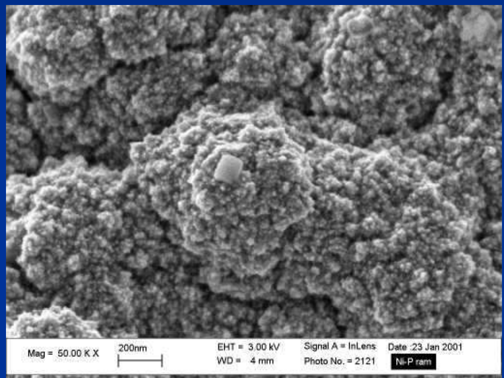
Micro-CT image of a snow pack



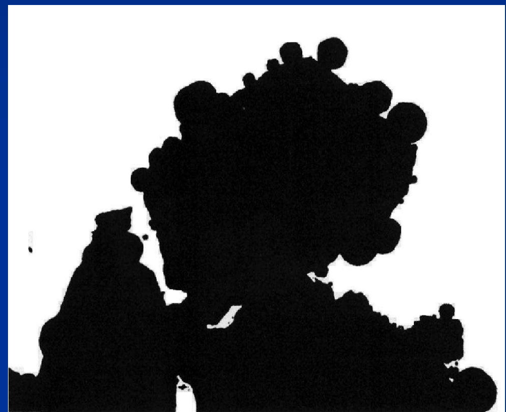
3D model of Vycor glass

Irregular shapes…

in chemistry and electrochemistry

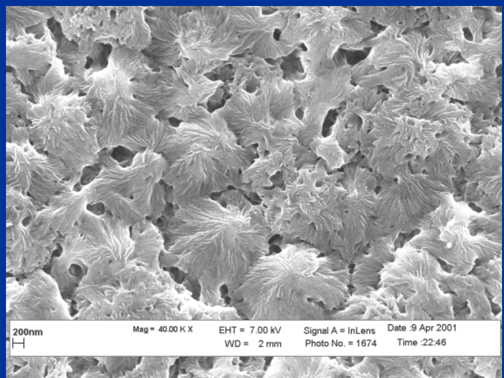


Microroughness of a nickel surface

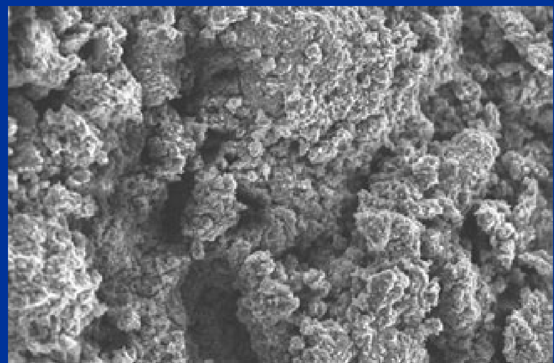


Typical shape of a porous catalyst

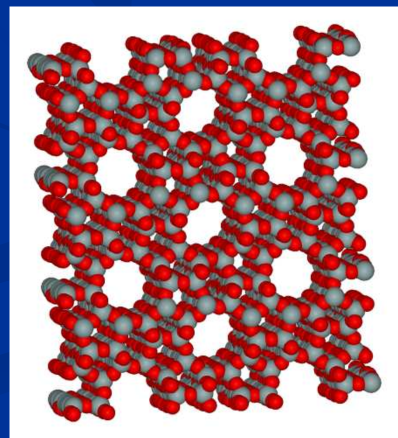
Copper dendrites



Nylon surface



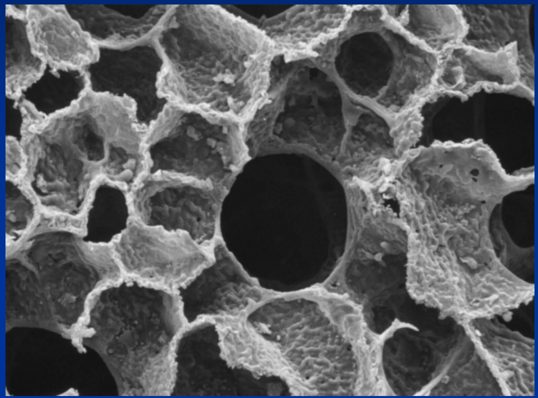
Porous polymer



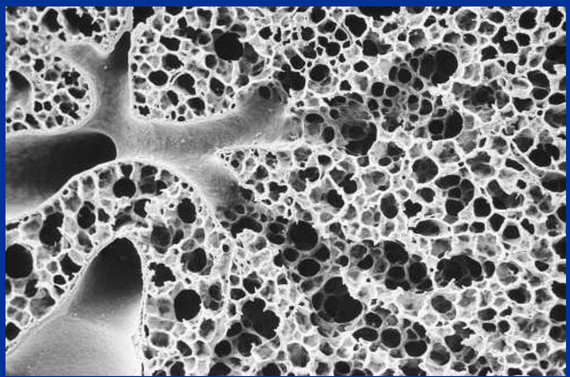
Zeolites

Irregular shapes…

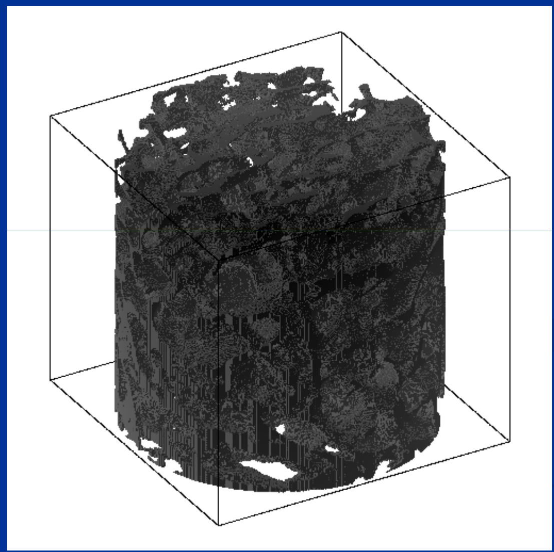
in physiology



Lung acinus

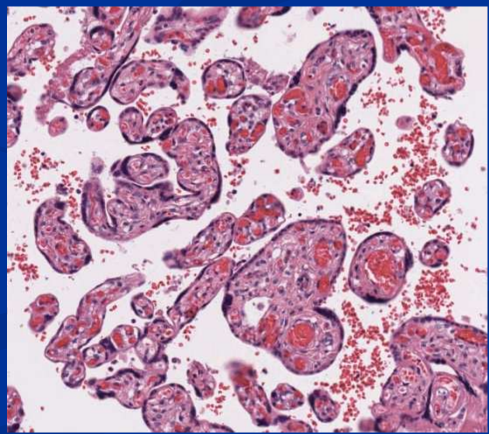


2D cut of a human acinus



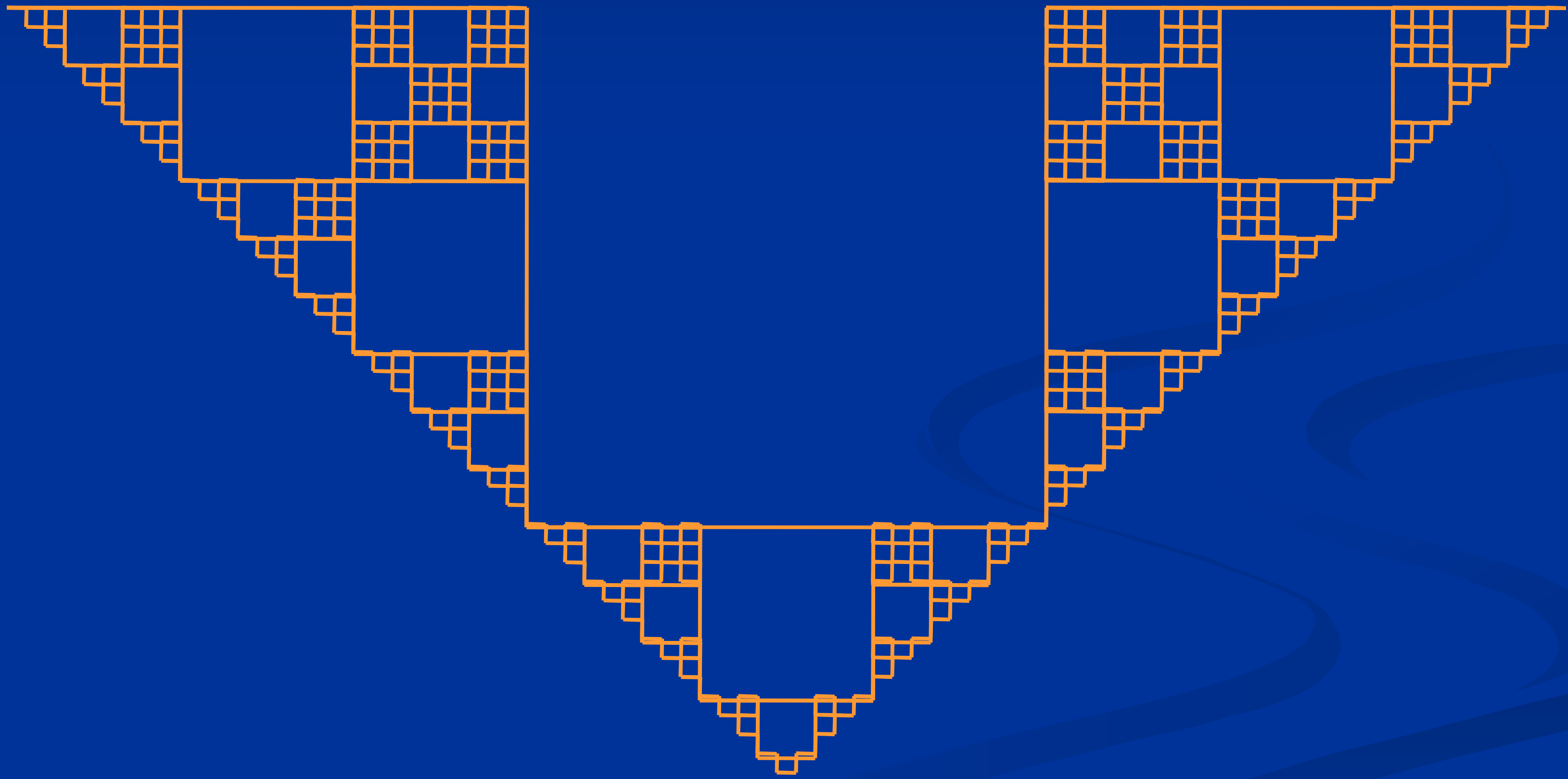
3D tomography of a bone

2D cut of a skin

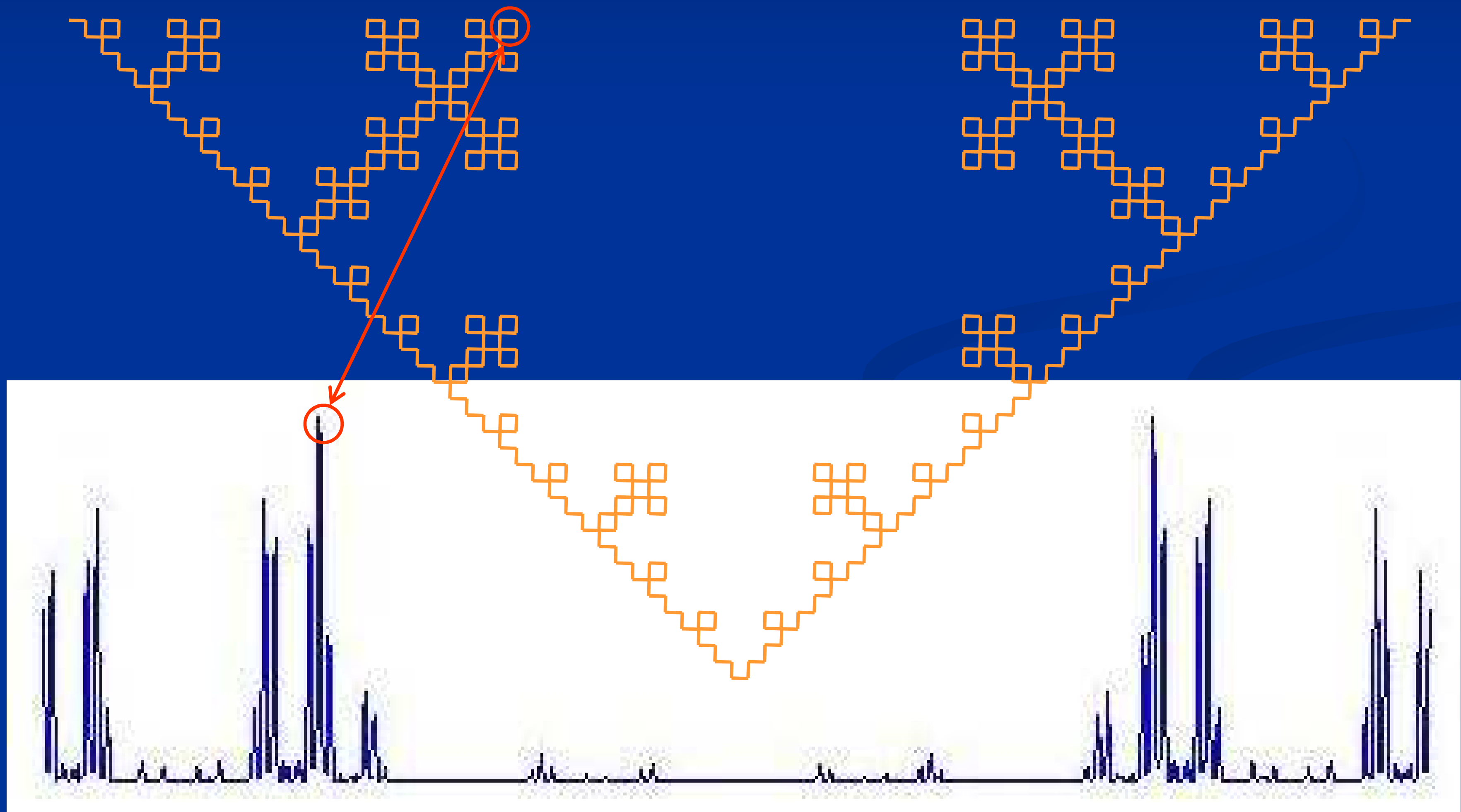


2D cut of a human placenta

Fractals: Von Koch curves

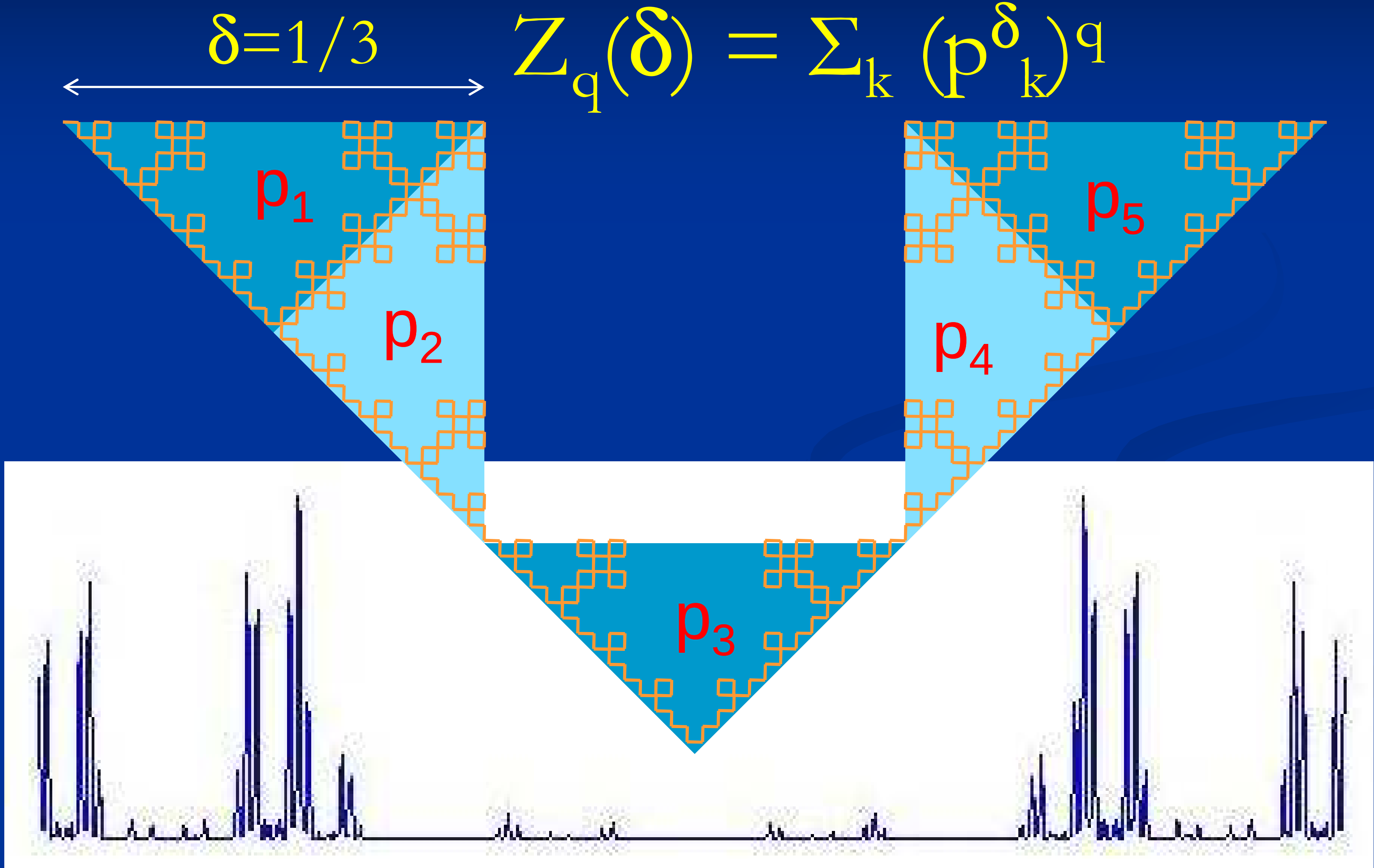


Fractals: Von Koch curves



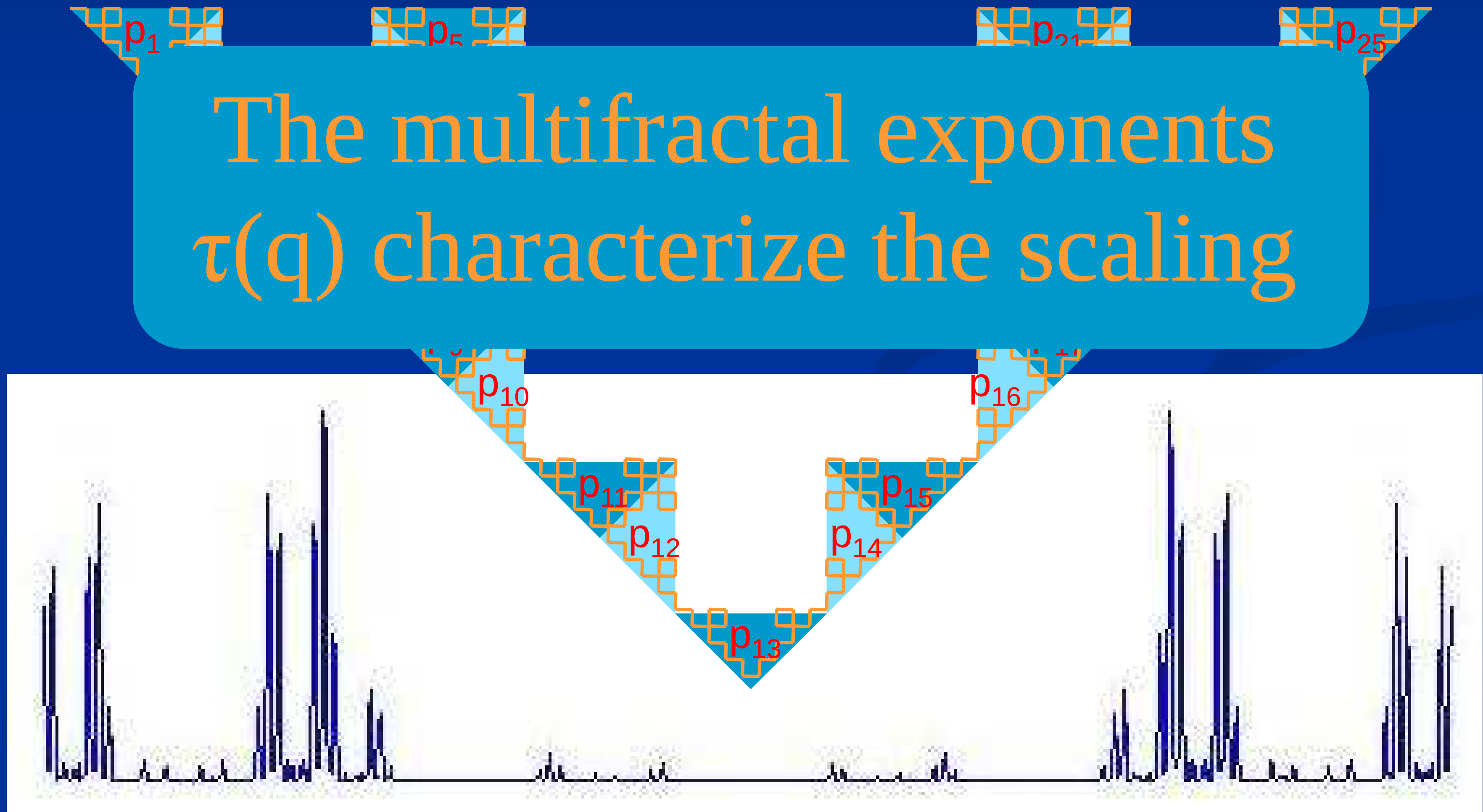
Multifractal analysis

$$Z_q(\delta) = \Sigma_k \, (p^{\delta}_{k})^q$$



Multifractal analysis

$\delta = 1/9$ $\qquad Z_q(\delta) = \Sigma_k \, (p^{\delta}_{k})^q \;\sim\; \delta^{\tau(q)}$



The multifractal exponents $\tau(q)$ characterize the scaling

Qualitative arguments

The moments $Z_q(\Lambda,\varepsilon,\delta)$ involves three lengths:

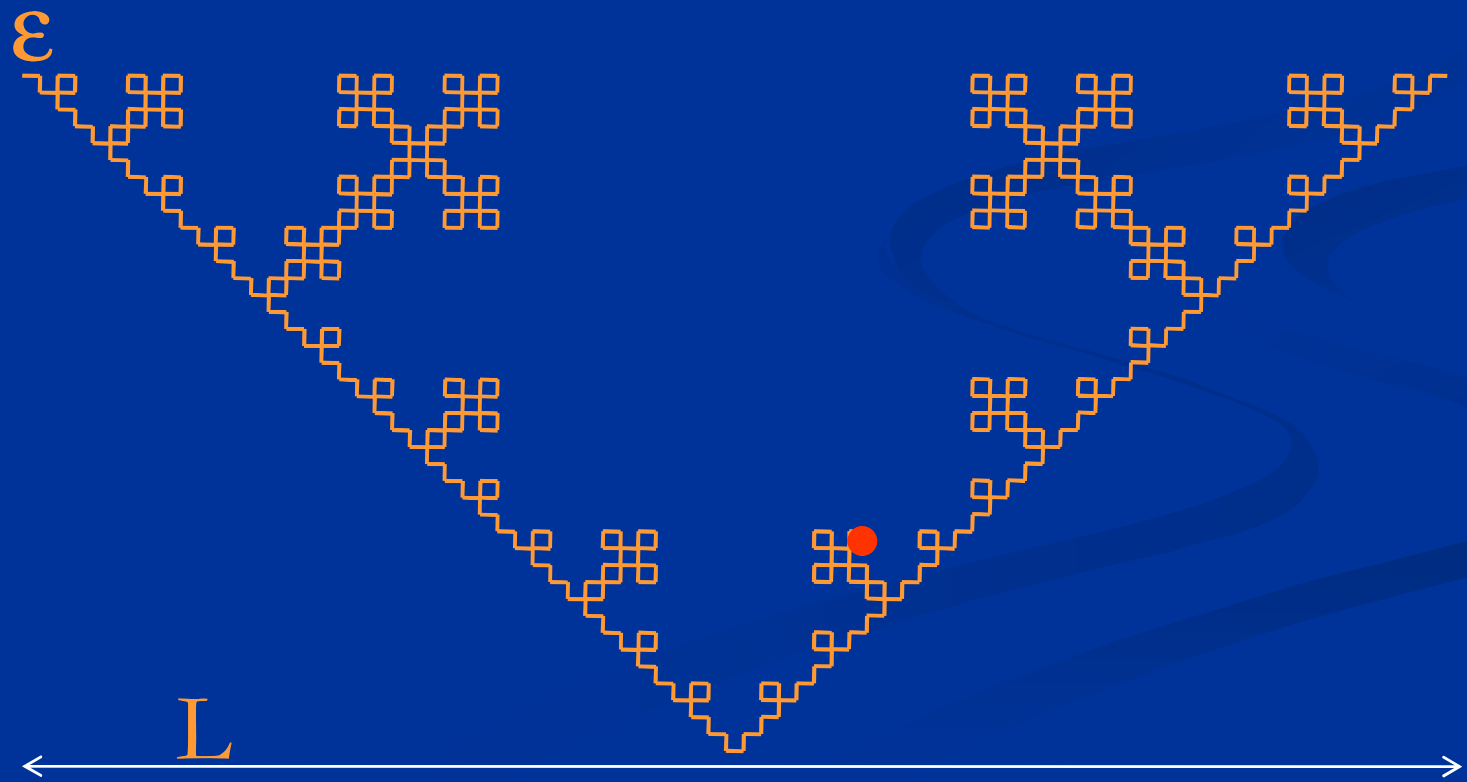
Λ the physical length (=D/W)

ε the minimal cut-off of prefractal boundary

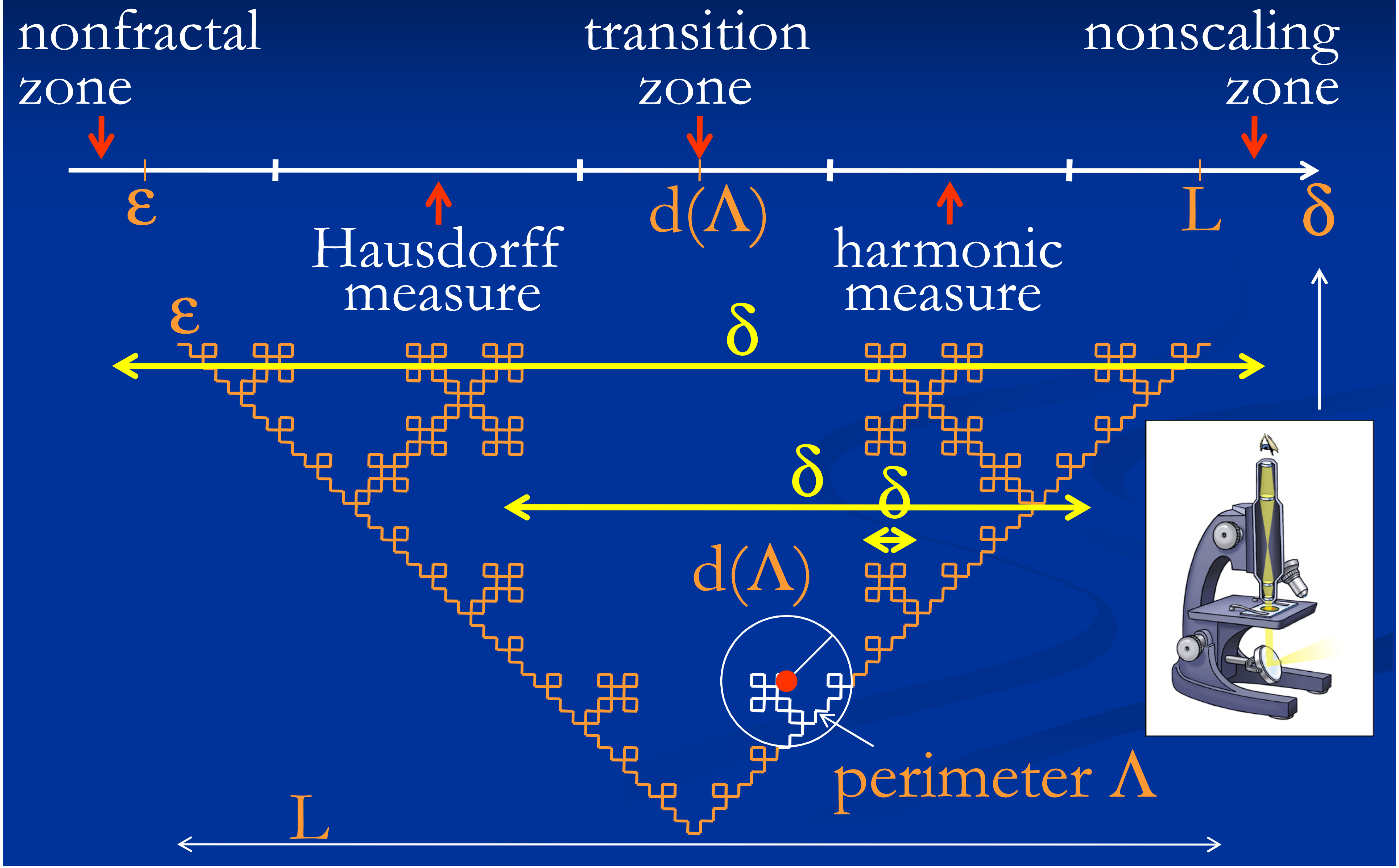
δ the scale at which the measure is studied

Let $d(\Lambda)$ be the diameter of a region of perimeter Λ which scales as $d(\Lambda) \sim \varepsilon(\Lambda/\varepsilon)^{1/Df}$

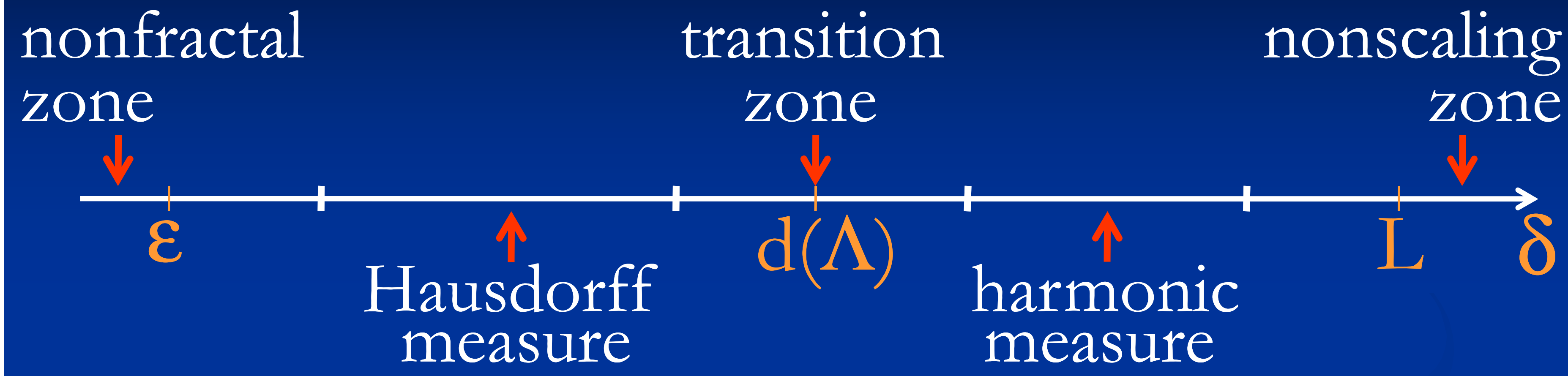
Qualitative arguments



Qualitative arguments



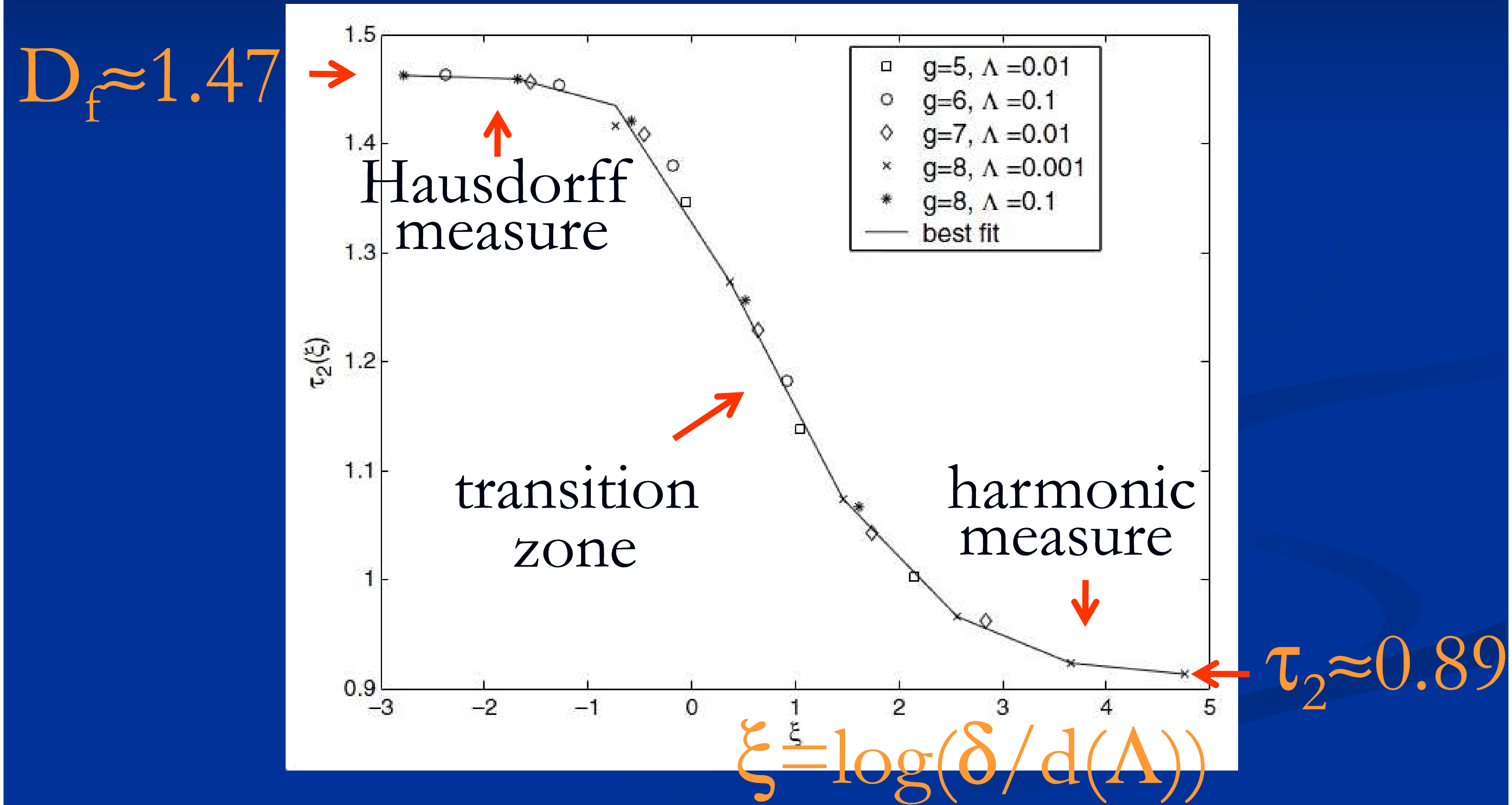
Qualitative arguments



$$\tau_q(\Lambda,\varepsilon,\delta)\approx D_f(q-1) \qquad \tau_q(\Lambda,\varepsilon,\delta)\approx\tau_q^{HM}$$

$$\tau_q(\Lambda,\varepsilon,\delta)=\tau_q(\delta/d(\Lambda))$$

Numerical verification



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Towards irregularity…

For a prefractal curve (cut-off ε):

$$\tau_q = \tau_q(\xi)$$

$$\xi = \log(\delta / d(\Lambda))$$

$$d(\Lambda) = \varepsilon(\Lambda/\varepsilon)^{1/Df}$$

When $\varepsilon \to 0$, $d(\Lambda) \to 0$ and $\tau_q(\xi) \to \tau_q^{HM}$

Trivial limit!

Can one get a nontrivial limit?

Conjectural extension

For a prefractal curve (cut-off ε):

$$\tau_q = \tau_q(\xi)$$

$$\xi = \log(\delta / d(\Lambda))$$

$$d(\Lambda) = \varepsilon(\Lambda/\varepsilon)^{1/Df}$$

We want $d(\Lambda)$ to be independent of ε…

$$\sigma \sim \varepsilon/\Lambda \quad \Rightarrow \quad \sigma \sim (\varepsilon/\Lambda)^{Df}$$

"Absorption perimeter" $\sim \varepsilon\, \sigma = \varepsilon(\varepsilon/\Lambda)^{Df}$, so Λ is now the diameter, and $\xi = \log(\delta/\Lambda)$

Potential applications

- Description of the transfer across partially active smooth or irregular interfaces
- Explanation of the constant-phase angle behavior in the impedance spectroscopy
- Selective filters or reactors that would depend on the diffusion coefficient (size)

Open mathematical problems

- How to define partially reflected Brownian motion and spread harmonic measures for truly irregular shapes?

- How to prove (empirical) scaling relations?

- What are the properties of the local time on irregular shapes?

Thank you for your attention !!!

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