Non asymptotic bounds for the distribution of the maximum of Random fields

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Signal + noise model

Spatial Statistics often uses “ signal + noise model”, for example :

- precision agriculture
- neuro-sciences
- sea-waves modelling

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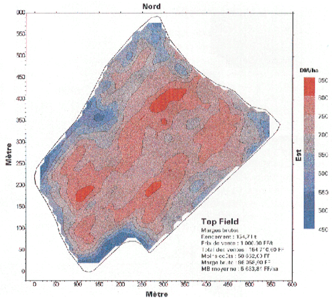
Precision agriculture

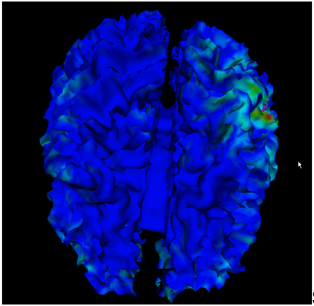
Representation of the yield per unit by GPS harvester .



Is there only noise or some region with higher fertility ? ?

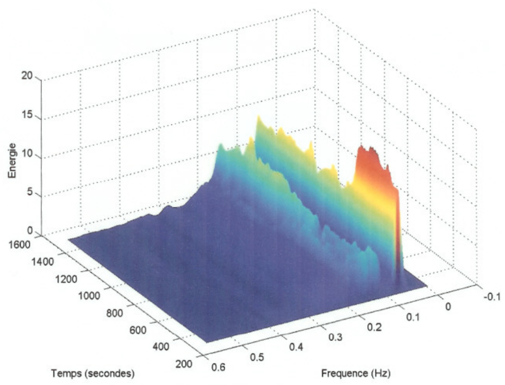
Neuroscience

The activity of the brain is recorded under some particular action and the same question is asked


source : Maureen CLERC

Sea-waves spectrum

Locally in time and frequency the spectrum of waves is registered.
We want to localize transition periods.



In all these situations a good statistics consists in observing the maximum of the (absolute value) of the random field for deciding if it is typically large (Noise) of not (signal).

1. Examples
2. The record method
3. The maxima method
   - Second order

Hypothesis

S is a regular set of $\mathbb{R}^2$ (compact, simply connected + piecewise $\mathcal{C}^1$ parametrization of the boundary by arc length) X is such that
-the bivariate process $Z = (X, \frac{\partial X}{\partial t_2})$ has $\mathcal{C}^1$ sample paths and non degenerated Gaussian distribution.
- the distribution of $(X(t), X'(t))$, $\left(X(t), \frac{\partial X}{\partial t_1}, \frac{\partial^2 X}{\partial t_2^2}\right)$ do not degenerate

Then Roughly speaking the event

$$\{M > u\}$$

is almost equal to the events

"The level curve at level$u$ is not empty"

"The point at the southern extremity of the level curve exists"

"There exists a point on the level curve :
$X(t) = u; X_{10}(t) = \frac{\partial X}{\partial t_1} = 0; X_{01}(t) = \frac{\partial X}{\partial t_2} > 0$
$X_{20}(t) = \frac{\partial^2 X}{\partial t_1^2} < 0$"

The computation of a probability is bounded by the expectation of the number of roots of the process $Z(t) - (u, 0)$ that can be computed by means of a Rice formula. Reintroducing the boundary to get an exact statement, we get

$$\mathbb{P}\{M > u\} \leq \mathbb{P}\{Y(0) > u\} + \int_0^L \mathrm{E}(|Y'(\ell)|) \big| Y(\ell) = u) p_{Y(\ell)}(u) d\ell$$

$$+ \int_S \mathrm{E}(|\det(Z'(t) \ \mathbb{1}_{X_{20}(t)<0} \ \mathbb{1}_{X_{01}(t)>0}| \big| X(t) = u, X_{01}(t) = 0) p_{X(t),X_{01}(t)}(u, 0) dt,$$

Where $Y(\ell)$ is the process X on the boundary and $\ell = 0$ corresponds to the southern extremity. The difficulty lies in the computation of the expectation of the determinant

The key point is that under the condition $\{X(t) = u, X_{01}(t)\} = 0$, the quantity

$$|\det(Z'(t)| = \begin{vmatrix} X_{10} & X_{01} \\ X_{11} & X_{02} \end{vmatrix}$$

is simply equal to $|X_{10}X_{02}|$. Taking into account conditions, we get the following expression for the second integral

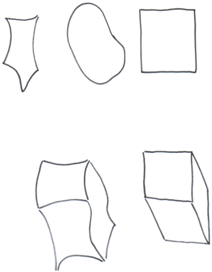
$$+\int_S \mathrm{E}(|X_{20}(t)^- X_{01}(t)^+ \big| X(t) = u, X_{01}(t) = 0) p_{X(t),X_{01}(t)}(u,0)dt.$$

Moreover under stationarity or some more general hypotheses, these two random variables are independent.

Consider a realization with $M > u$, then necessarily there exist a local maxima or a border maxima above U

Border maxima : local maxima in relative topology

If the consider sets with polyedral shape union of manifold of dimension 1 to d



In fact result are simpler (and stronger) in term of the density $p_M(x)$ of the maximum. Bound for the distribution are obtained by integration.

Theorem

$$p_M(x) \leq \widehat{p}_M(x) := \frac{1}{2}[\overline{p}_M(x) + p_M^{EC}(x)] with$$

$$\overline{p}_M(x) := \int_S \mathrm{E}\big(|\det(X"(t))|/X(t) = x, X'(t) = 0\big) p_{X(t),X'_j(t)}(x,0)dt + \textit{boundary term}$$

and

$$p_M^{EC}(x) := (-1)^d \int_S \mathrm{E}\big(\det(X"(t))/X(t) = x, X'(t) = 0\big) p_{X(t),X'_j(t)}(x,0)dt + \textit{boundary}$$

Quantity $p_M^{EC}(x)$ is easy to compute using the work by Adler and properties of symmetry of the order 4 tensor of variance of $X"$ (under the conditional distribution))

Lemma

$$\mathrm{E}\Big(\det(X"(t))/X(t) = x, X'(t) = 0\Big) = \det(\Lambda)H_d(x)$$

where $H_d(x)$ *is the* d*th Hermite polynomial and* $\Lambda := \mathrm{Var}(X'(t))$

main advantage of Euler characteristic method lies in this result.

computation of $\bar{p}_m$

The key point is the following
If X is stationary and isotropic with covariance $\rho(\|t\|^2)$ normalized by $\mathrm{Var}(X(t)) = 1$ et $\mathrm{Var}(X'(t)) = Id$
Then under the condition $X(t) = x, X'(t) = 0$

$$X"(t) = \sqrt{8\rho"}G + \xi\sqrt{\rho" - \rho'^2}Id + xId$$

Where G is a GOE matrix (Gaussian Orthogonal Ensemble), and ξ a standard normal independent variable. We use recent result on the the characteristic polynomials of the GOE. Fyodorov(2004)

Theorem

Assume that the random field $\mathcal{X}$ is centered, Gaussian, stationary and isotrpic and is "regular" Let S have polyhedral shape. Then,

$$\overline{p}(x) = \varphi(x)\left\{\sum_{t\in S_0}\widehat{\sigma}_0(t) + \sum_{j=1}^{d_0}\left[\left(\frac{|\rho'|}{\pi}\right)^{j/2}H_j(x) + R_j(x)\right]g_j\right\} \qquad (1)$$

- $g_j = \int_{S_j}\widehat{\theta}_j(t)\sigma_j(dt)$, $\widehat{\sigma}_j(t)$ *is the normalized solid angle of the cone of the extended outward directions at t in the normal space with the convention $\sigma_d(t) = 1$.*
  For convex or other usual polyhedra $\widehat{\sigma}_j(t)$ *is constant for $t \in S_j$,*
- H_j *is the j th(probabilistic) Hermite polynomial.*

Theorem (continued)

- $R_j(x) = \left(\frac{2\rho''}{\pi|\rho'|}\right)^{\frac{j}{2}} \frac{\Gamma((j+1)/2}{\pi} \int_{-\infty}^{+\infty} T_j(v) \exp\left(-\frac{y^2}{2}\right) dy$

$$v := -(2)^{-1/2}\left((1-\gamma^2)^{1/2}y - \gamma x\right) \text{ with } \gamma := |\rho'|(\rho'')^{-1/2}, \tag{2}$$

$$T_j(v) := \left[\sum_{k=0}^{j-1} \frac{H_k^2(v)}{2^k k!}\right] e^{-v^2/2} - \frac{H_j(v)}{2^j (j-1)!} I_{j-1}(v), \tag{3}$$

$$I_n(v) = 2e^{-v^2/2} \sum_{k=0}^{[\frac{n-1}{2}]} 2^k \frac{(n-1)!!}{(n-1-2k)!!} H_{n-1-2k}(v) \tag{4}$$

$$+ \mathbb{I}_{\{n \text{ even}\}}\, 2^{\frac{n}{2}} (n-1)!! \sqrt{2\pi}(1-\Phi(x))$$

Second order study

Using an exact implicit formula

Theorem

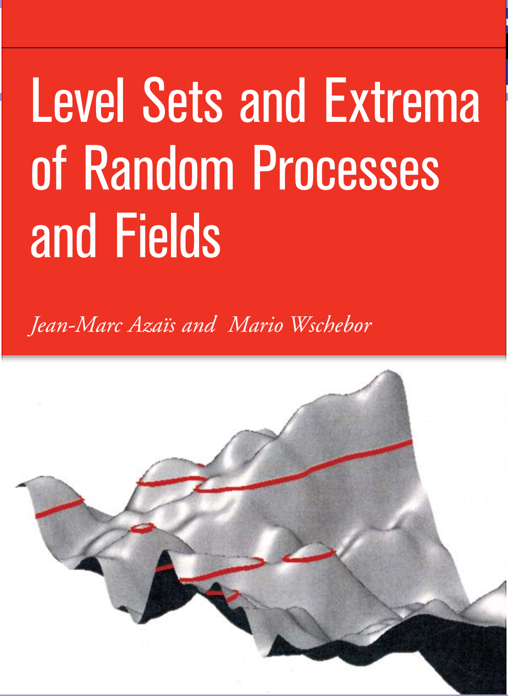
Under conditions above + $\mathrm{Var}(X(t)) \equiv 1$ *Then*

$$\underline{\lim}_{x\to+\infty} \; -\frac{2}{x^2}\log\left[\widehat{p}_M(x) - p_M(x)\right] \;\geq\; 1 + \inf_{t\in S}\frac{1}{\sigma_t^2 + \overline{\lambda}(t)\kappa_t^2}$$

$$\sigma_t^2 := \sup_{s\in S\setminus\{t\}} \frac{\mathrm{Var}\big(X(s)/X(t), X'(t)\big)}{(1-r(s,t))^2}$$

and κ_t *is some geometrical characteristic et* $\Lambda_t = GEV(\Lambda(t))$

The right hand side is finite and > 1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Jean-Marc Azaïs and Mario Wschebor

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