Testing Fractal Connectivity in Multivariate Long Memory Processes

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MultiVariate Data ?

- Applications (Biology, Medecine, Environment, Telecomm.) :
  - large systems,
  - out of equilibrium,
  - dynamical.
- Analyses :
  - numerous censors,
  - heterogeneous censors,

⇒ MultiVariate (MultiComponent) Data
  - how much information shared between component ?
  - correlated and redundant information ?

- Long memory (long range dependence) :
  - slow (algebraic) decrease of correlations in time,
  - LM observed independently on each Component,
  - a same and single phenomenon ?
  - or different causes for LM ?

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Long Memory

- Long Memory :
  - Y 2nd order stationary process :

    Spectrum : $\Gamma_Y(f) \sim C_\Gamma |f|^{-\gamma}$, $|f| \to 0$, $0 < \gamma < 1$,
    Covariance : $\gamma_Y(\tau) \sim C_\gamma |\tau|^{-(1-\gamma)}$, $|\tau| \to \infty$
- Self-similarity :
  - H-ss : $\{X(t), t \in \mathcal{R}\} \stackrel{fdd}{=} \{a^H X(t/a), t \in \mathcal{R}\}, \ \forall a > 0$

    if X is H-ss,
    with stationary increments, $Y(k) = X(k+1) - X(k)$
    if $1/2 < H < 1$,
    then Y is Long Memory with $\gamma = 2H - 1$.
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Bivariate Long Memory

- $\mathbf{Z} = \{\mathbf{Z}(t)\}_{t\in\mathbb{Z}} = \{[X(t), Y(t)]\}_{t\in\mathbb{Z}}$
  bivariate long memory process iff
  $N-$th order increment process : $\tilde{\mathbf{Z}}(t) = \delta^N \mathbf{Z}(t)$:
  - is stationary,
  - has spectral and inter-spectral densities ($-\pi \le f \le \pi$) :

$$\begin{aligned}
\Gamma_{\tilde{X}}(f) &= \Omega_X \left|1 - e^{-jf}\right|^{-2\alpha_X} \Gamma^*_X(f) \\
\Gamma_{\tilde{Y}}(f) &= \Omega_Y \left|1 - e^{-jf}\right|^{-2\alpha_Y} \Gamma^*_Y(f) \\
\Gamma_{\tilde{X}\tilde{Y}}(f) &= \Omega_{12} \left(1 - e^{-jf}\right)^{-\alpha_1} \left(1 - e^{jf}\right)^{-\alpha_2} \Gamma^*_{12}(f).
\end{aligned}$$

  - $\Omega_\cdot$ positive multiplicative factors,
  - $\alpha_{(\cdot)} \in [0, 0.5]$,
  - $\Gamma^*_{(\cdot)}(f)$ non-negative, symmetric,
    with limit 1 at $|f| \to 0$, hence modeling *short memory*.

Fractal Connectivity

- Coherence Function :

$$C_{\tilde{X}\tilde{Y}}(f) = \frac{|\Gamma_{\tilde{X}\tilde{Y}}(f)|}{\sqrt{\Gamma_{\tilde{X}}(f)\Gamma_{\tilde{Y}}(f)}}$$

- Bivariate long memory :

$$C_{\tilde{X}\tilde{Y}}(f) \sim_{f\to 0} C_0|f|^{-(\alpha_{XY}-\alpha_X-\alpha_Y)}$$
with $\alpha_{XY} = \alpha_1 + \alpha_2$,
with $C_0 = |\Omega_{12}|/\sqrt{\Omega_{\tilde{X}}\Omega_{\tilde{Y}}}$.

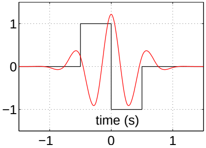
- Positive definite :

$$0 \leq C_{\tilde{X}\tilde{Y}}(f) \leq 1 \Rightarrow \alpha_{XY} \leq \alpha_X + \alpha_Y.$$

- Fractal Connectivity :

$$\boxed{C_0 \neq 0 \text{ and } \alpha_{XY} = \alpha_X + \alpha_Y.}$$

Wavelets and Long Memory



- Wavelets :
  - Mother-Wavelet ψ : Oscillating pattern,
  - Number of vanishing moments N_ψ :

    $\forall k = 0, ..., N-1$,
    $\int_{\mathcal{R}} t^k \psi_0(t) dt \equiv 0$ and $\int_{\mathcal{R}} t^N \psi_0(t) dt \neq 0$.
  - Basis : $\{\psi_{j,k}(t) = 2^{-j/2}\psi_0(2^{-j}t - k)\}$,
  - Coefficients of Y : $d_Y(j,k) = \langle \psi_{j,k}, Y \rangle$

- Wavelets and 2nd order bivariate stationary processes :
  - $\mathbb{E}|d_X(j,k)|^2 = \int_{\mathcal{R}} \Gamma_X(f) 2^j |\tilde{\Psi}_0(2^j f)|^2 df$,

- Wavelets and LRD :
  - $\mathbb{E}|d_Y(j,k)|^2 \sim C 2^{j(2H-1)}$ for $2^j \to +\infty$,
  - $S(j) = \frac{1}{n_j}\sum_k |d_Y(j,k)|^2$,
  - Logscale Diagram : $\log_2 S(j)$ vs $\log_2 2^j = j$

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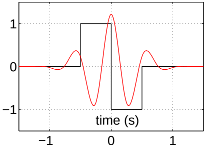

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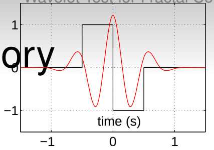

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  - Logscale Diagram : $\log_2 S(j)$ vs. $\log_2 2^j = j$,
  - $\hat{H} = \frac{1}{2}\left(1 + \sum_{j=j_1}^{j_2} w_j \log_2 S(j)\right)$.

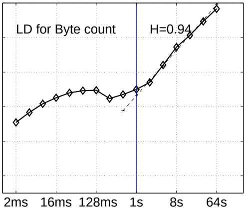
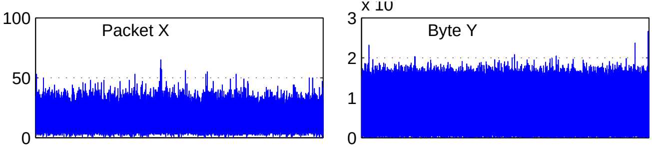


Illustration : Internet Traffic

- Aggregated Count Times Series :



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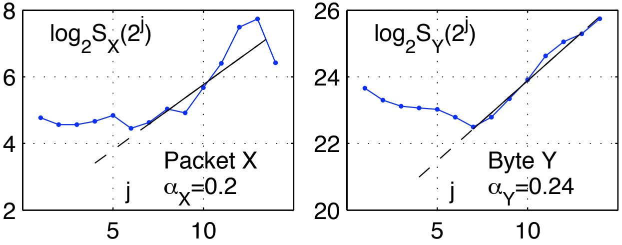

- A single Long Memory or different phenomena ?

Wavelets and bivariate Long Memory

- Wavelets and 2nd order bivariate stationary processes :
  - $\mathbf{E}|d_{\tilde{X}}(j,k)|^2 = \int_{\mathcal{R}} \Gamma_{\tilde{X}}(f) 2^j |\tilde{\Psi}_0(2^j f)|^2 df,$
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  - $\mathbf{E} d_{\tilde{X}}(j,k) d_{\tilde{Y}}(j,k) = \int_{\mathcal{R}} \Gamma_{\tilde{X},\tilde{Y}}(f) 2^j |\tilde{\Psi}_0(2^j f)|^2 df.$

- Wavelets and bivariate Long Memory :

  if $N_\psi > N$, and when $2^j \to +\infty$,
  - $\mathbf{E}|d_{\tilde{X}}(j,k)|^2 \sim c_X 2^{2j(\alpha_X + N)},$
  - $\mathbf{E}|d_{\tilde{Y}}(j,k)|^2 \sim c_X 2^{2j(\alpha_Y + N)},$
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  $\gamma_{XY}(2^j) = \mathbf{E} d_X(j,k) d_Y(j,k) / \sqrt{\mathbf{E} d_X(j,k)^2 \mathbf{E} d_Y(j,k)^2}$
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Wavelets and Fractal Connectivity : Estimation

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$$\widehat{\mathbf{E}|d_X(j,k)|^2} = S_X(2^j) = \frac{1}{n_j}\sum_{k=1}^{n_j} d_X(j,k)^2,$$
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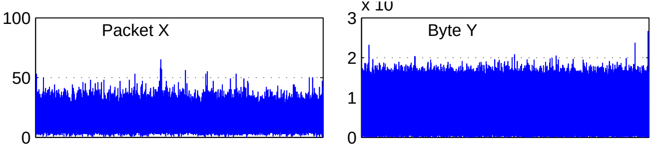

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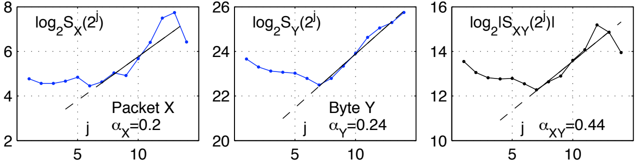

- A single Long Memory or different phenomena ?

$$\hat{\alpha}_{XY} = \hat{\alpha}_X + \hat{\alpha}_Y \, ? \qquad \widehat{\alpha_{XY} - \alpha_X - \alpha_Y} = 0 \, ?$$

Wavelet Test for Fractal Connectivity

- Fisher $z-$transform :

$$\hat{z}_{XY}(2^j) = \frac{1}{2}\ln\frac{1+\hat{\gamma}_{XY}(2^j)}{1-\hat{\gamma}_{XY}(2^j)} \overset{d}{\sim} \mathcal{N}(z_{XY}(2^j), \sigma(2^j)),$$
$$z_{XY}(2^j) = \frac{1}{2}\ln\frac{1+\gamma_{XY}(2^j)}{1-\gamma_{XY}(2^j)},$$
$$\sigma^2(2^j) = \frac{1}{n_j-3}.$$

- Null Hypothesis :

$$H_0 : z_{XY}(2^{J_1}) \equiv z_{XY}(2^{J_1+1}) \equiv \cdots \equiv z_{XY}(2^{J_2})$$
$$j \in [J_1, J_2], J = J_2 - J_1 + 1.$$

- Test Statistics :

$$\hat{V}_J = \sum_{j=J_1}^{J_2} \frac{1}{\sigma^2(j)}\left(\hat{z}_{XY}(2^j) - \frac{\sum_{j=J_1}^{J_2}\hat{z}_{XY}(2^j)/\sigma^2(2^j)}{\sum_{j=J_1}^{J_2} 1/\sigma^2(2^j)}\right)^2$$

- Test Formulation :

Under H_0, $\hat{V}_J \sim \chi^2_{(J-1)}$,
Significance Level : α,
Test : $\hat{d}_J = 1$ if $\hat{V}_J > C_\chi(\alpha)$, $\hat{d}_J = 0$ otherwise.

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Test Statistics : Illustration on Synthetic Data :

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  Under H_1, $\hat{V}_J \sim \chi^2_{(J-1),V_J}$, Non central χ^2 with

$$V_J = \sum_{j=J_1}^{J_2} \frac{1}{\sigma^2(2^j)} \left(z_{XY}(2^j) - \frac{\sum_{j=J_1}^{J_2} z_{XY}(2^j)/\sigma^2(2^j)}{\sum_{j=J_1}^{J_2} 1/\sigma^2(2^j)} \right)^2$$

- Monte Carlo Simulations : 1024 realizations,

Test Statistics : Illustration on Synthetic Data :

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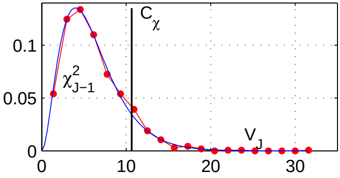


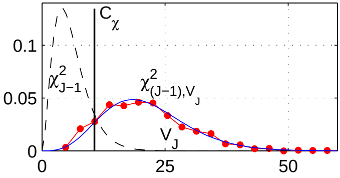


Illustration on Synthetic Data : H_0

- Under H_0, no short-memory :

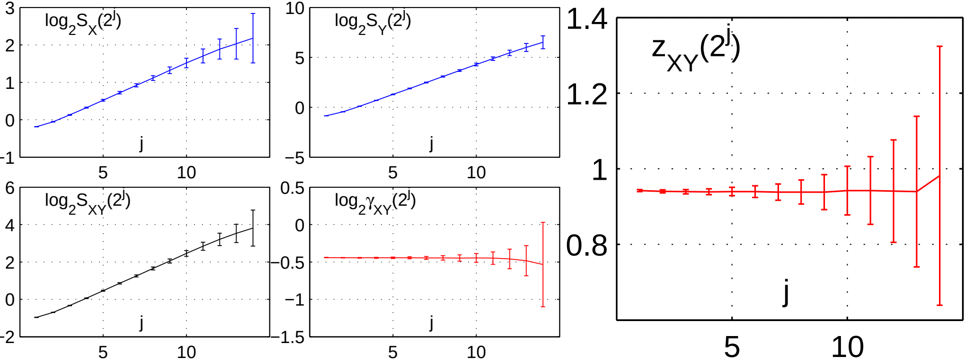
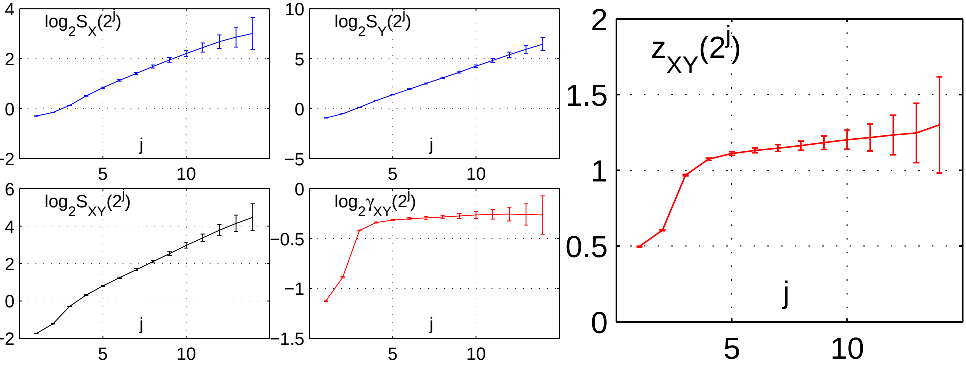


Illustration on Synthetic Data : H_0

- Under H_0, with short-memory :



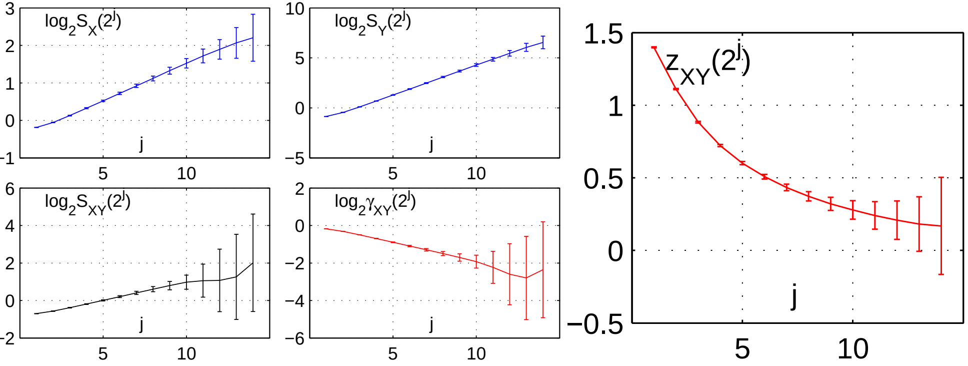
Performances on Synthetic Data

$\alpha_{XY} = 0.4$ - $\mathbf{[J_1, J_2] = [\cdot, 13]}$ - 10% significance							
Gaussian no short memory - $C_0 = 0.7$							
J_1	5	6	7	8	9	10	11
$\bar{d}_J$	11.0	9.6	9.1	8.6	9.3	9.0	8.5
$\bar{p}_J$	0.51	0.51	0.51	0.51	0.51	0.51	0.51
Gaussian ARMA(1,1) - $C_0 = 0.7$							
J_1	5	6	7	8	9	10	11
$\bar{d}_J$	66.7	33.5	21.5	14.4	11.2	8.9	9.2
$\bar{p}_J$	0.12	0.29	0.38	0.45	0.50	0.51	0.52

TAB.: **Scale range.** Mean test decisions (in %) and p-values for different values of J_1, without (top) and with (bottom) Short Memory.

Illustration on Synthetic Data : H_1

- Under H_1, no short-memory :



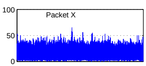
Performances on Synthetic Data

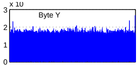
$[J_1, J_2] = [7, 13]$ - 10% significance					
Test decisions - mean rejection rates $\bar{d}_J$ (in %)					
α_{XY}	**0.4**	0.35	0.3	0.25	0.2
$C_0 = 0.5$	**9.4**	22.7	42.9	54.9	56.8
$C_0 = 0.7$	**10.3**	51.9	83.0	89.2	89.3
$C_0 = 0.9$	**8.7**	99.3	99.9	99.9	99.9
Mean p-value $\bar{p}_J$					
α_{XY}	**0.4**	0.35	0.3	0.25	0.2
$C_0 = 0.5$	**0.51**	0.38	0.25	0.18	0.16
$C_0 = 0.7$	**0.50**	0.18	0.06	0.04	0.04
$C_0 = 0.9$	**0.50**	0.00	0.00	0.00	0.00

TAB.: **Test performance.** Mean test decisions (top) and p-values (bottom) for different values of C_0 and α_{XY}.

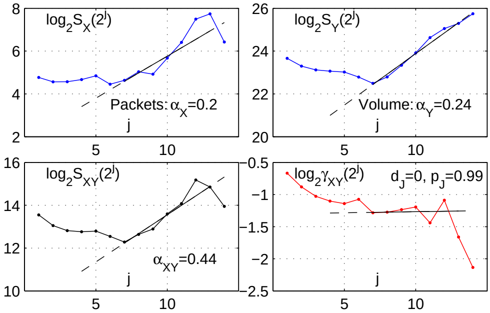
Illustration : Internet Traffic

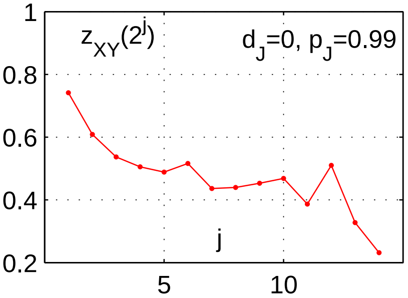
- Aggregated Count Times Series :





- No anomaly in Traffic :





  - p- value is high, test does not reject,
  - a single Long Memory in Bytes and Packets Counts,
- Anomalies in Traffic :
  - p- value is low, test rejects,
  - LM in Bytes and Packets Counts is no longer related.

Conclusions

- Fractal Connectivity ?
  - A single mechanism controls long memory on all components of multivariate data ?
- Long memory ?
  - Think wavelet !
- Test for Fractal Connectivity ?
  - Wavelet Coherence Function,
  - Fisher z-Transform,
- Internet Traffic ?
  - Long memory in Packets and Bytes are related,
  - Anomalies breaks the single mechanism.

- References :
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