From statistics to topology and back again

Robert Adler

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Industrial Engineering & Management

and

Electrical Engineering

–

Technion – Israel Institute of Technology

Paris: January, 2009

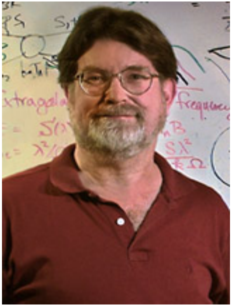
Back to the future

Early Development of the Universe

The Nobel Prize in Physics 2006

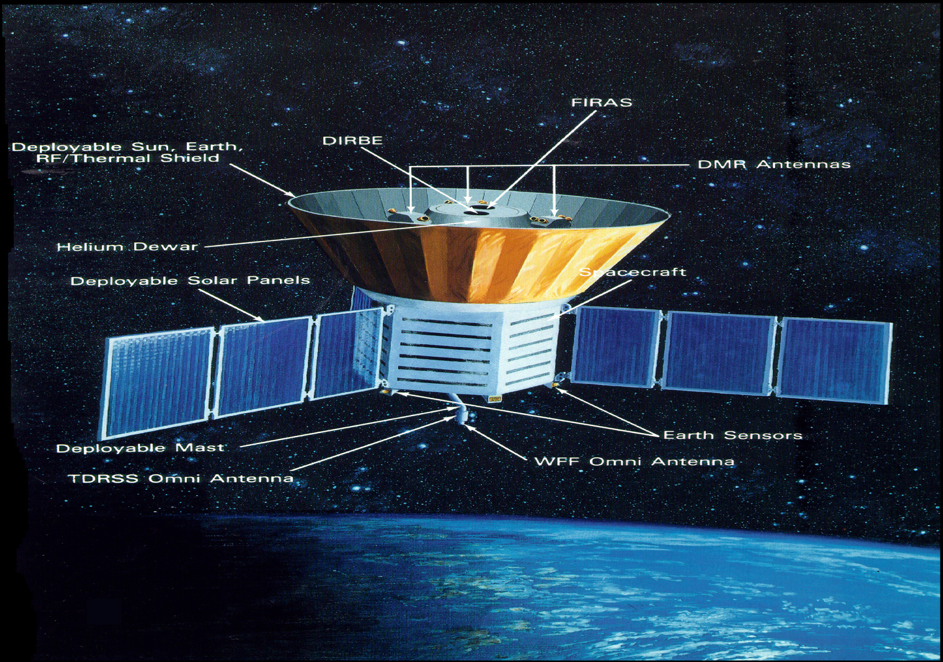
"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

John C. Mather
NASA
b. 1946

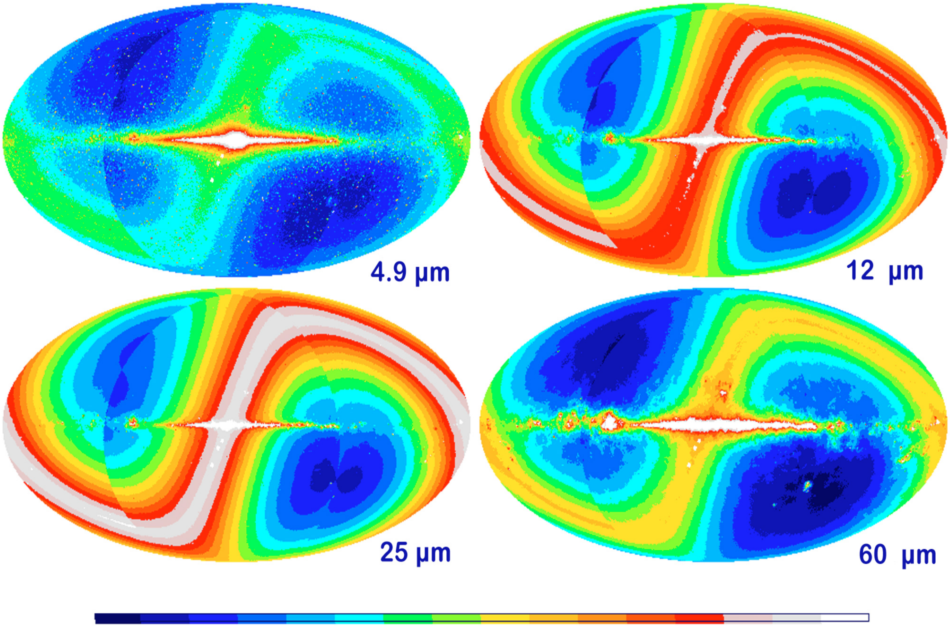


George F. Smoot
Berkeley
b. 1945

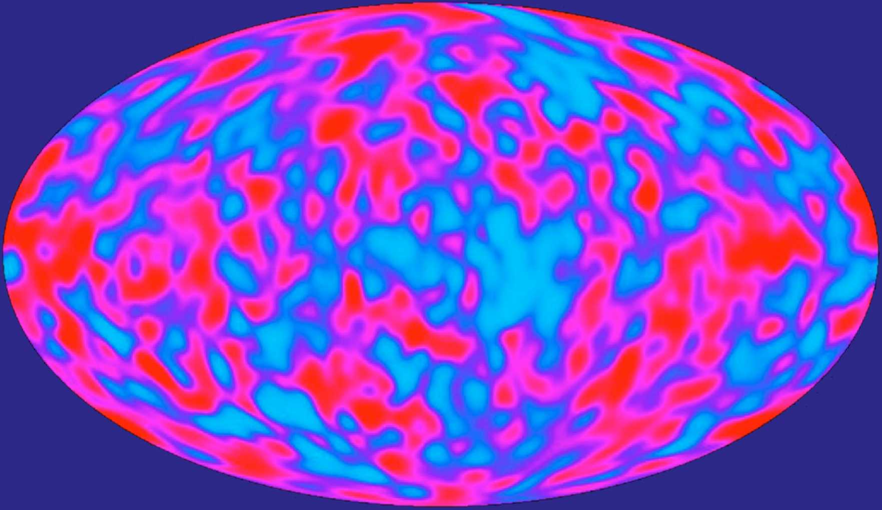
The COBE Satellite



DIRBE Solar Elongation 90° Maps: Mid-Infrared

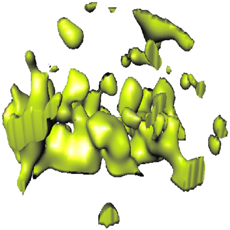


DMR's Two Year CMB Anisotropy Result



Center for Astrophysics (CfA) survey

10,506 galaxies in the cone-shaped survey region, which extends out to 135 megaparsecs in the northern hemisphere, with the earth at the apex of the cone.



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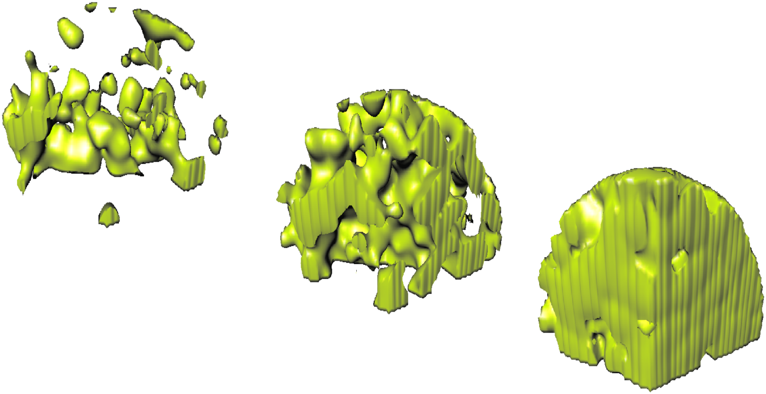
10,506 galaxies in the cone-shaped survey region, which extends out to 135 megaparsecs in the northern hemisphere, with the earth at the apex of the cone.



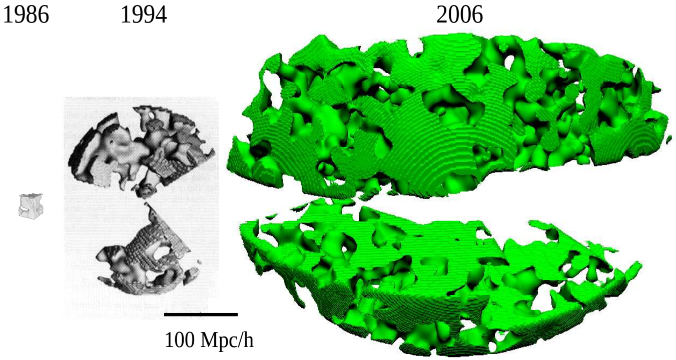
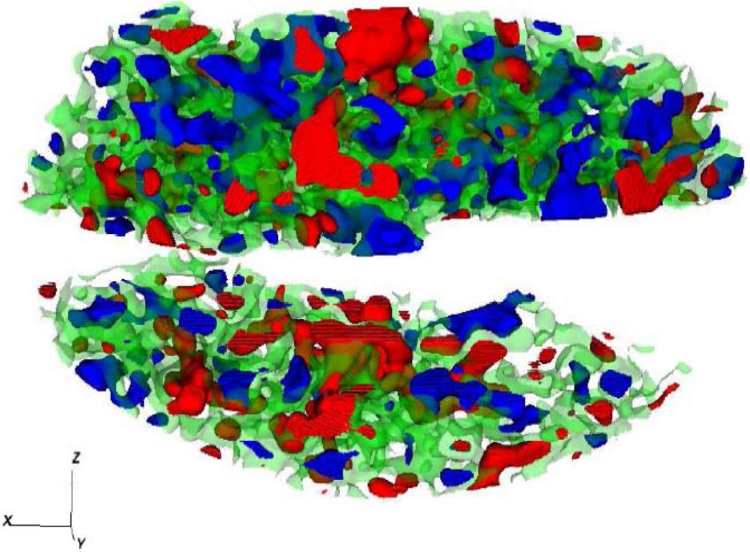
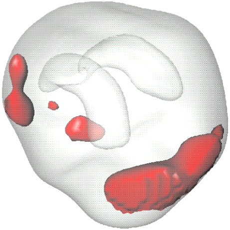
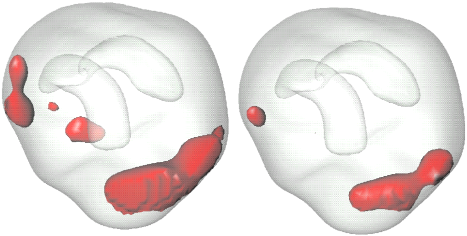


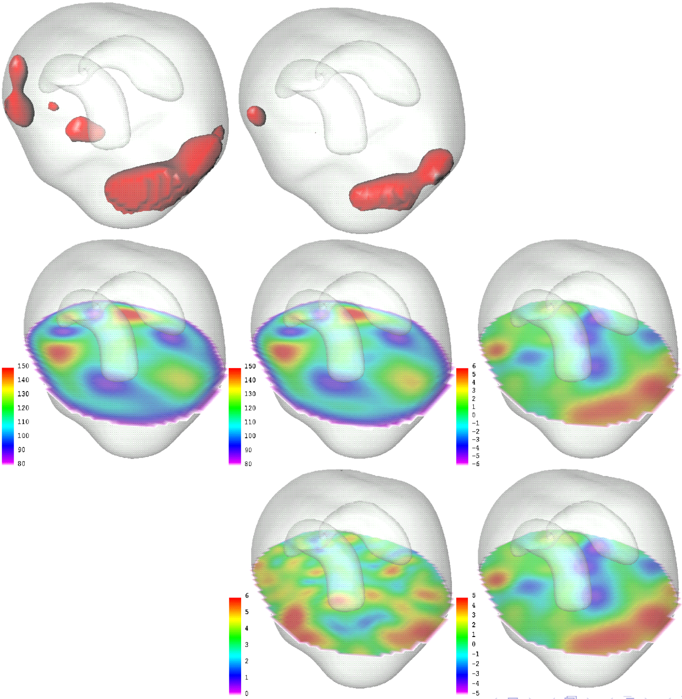
FIG. 1.— 50% high volume contours from three galaxy surveys across three decades. From left to right, they are Gott, Melott, & Dickinson (1986), Vogeley et al. (1994), and the present work.


Z
X
Y

Mapping the brain






150
140
130
120
110
100
90
80
150
140
130
120
110
100
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80
6
5
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3
2
1
0
-1
-2
-3
-4
-5
-6
6
5
4
3
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3
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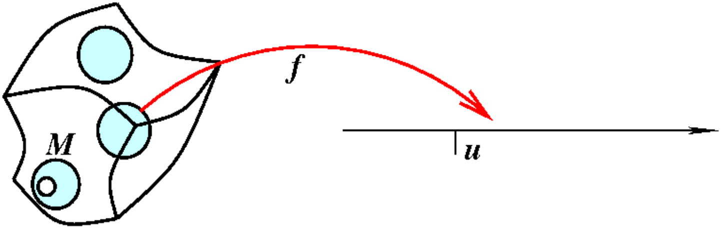
-1
-2
-3
-4
-5

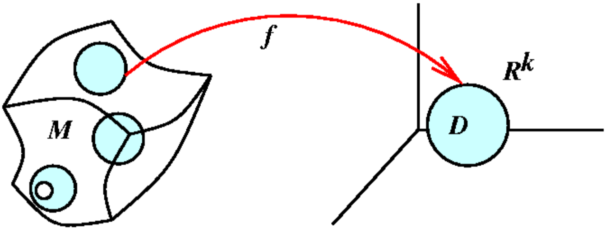
A general formulation

Excursion sets



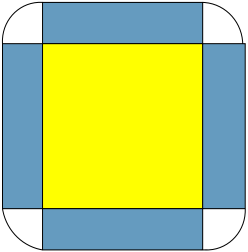
$$A_u \equiv A_u(f, M) \triangleq \{t \in M : f(t) \geq u\}$$

A more general setting



$$A_D \equiv A_D(f, M) \triangleq \{t \in M : f(t) \in D\}$$

Geometry

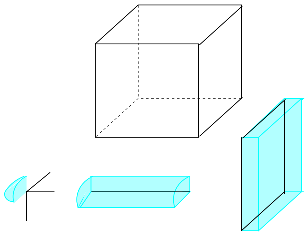


$$\lambda_2\left(\mathrm{Tube}(M,\rho)\right) \;=\; \pi\rho^2 \;+\; \rho\times 4L \;+\; L^2$$

$$\;=\; \sum_{j=0}^{2} \omega_{2-j}\rho^{2-j}\mathcal{L}_j(M)$$

where

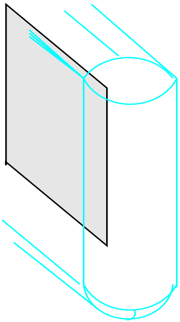
$$\omega_j \;=\; \frac{\pi^{j/2}}{\Gamma\left(\frac{j}{2}+1\right)} \;=\; \frac{s_j}{j}$$



$$\lambda_3\left(\mathrm{Tube}(M,\rho)\right)$$

$$= \frac{4}{3}\pi\rho^3 \; + \; 12\cdot\frac{1}{4}\pi\rho^2\cdot L \; + \; 6\rho L^2 \; + \; L^3$$

$$= \sum_{j=0}^{3}\omega_{3-j}\rho^{3-j}\mathcal{L}_j(M)$$



$$\lambda_3\left(\mathrm{Tube}(M,\rho)\right) \;=\; \sum_{j=0}^{2} \omega_{3-j}\rho^{3-j}\mathcal{L}_j(M)$$

Steiner's Formula

1796-1863, Switzerland

For nice (e.g. convex) $M \in R^N$, and $N' \geq N$, the volume of

$$\mathrm{Tube}(M,\rho) \; = \; \left\{ t \in R^{N'} : d_{N'}(t,M) \leq \rho \right\}$$

is, for $\rho < \rho_c(M)$, given by,

$$\lambda_{N'}\left(\mathrm{Tube}(M,\rho)\right) \; = \; \sum_{j=0}^{N} \omega_{N'-j}\rho^{N'-j}\mathcal{L}_j(M)$$

The $\mathcal{L}_j$ can be defined via the tube formula and are *intrinsic*.

$$\lambda_{N'}\left(\mathrm{Tube}(M,\rho)\right) = \sum_{j=0}^{N} \omega_{N'-j}\rho^{N'-j}\mathcal{L}_j(M)$$

Wilhelm Killing
Germany
1847-1923

Rudolf Lipschitz
Germany
1832-1903

Hermann Weyl
Germany/USA
1885-1951

Treat M as a Riemannian manifold

Curvature tensor:,

$$R(X, Y, Z, W) \;=\; \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\; W\right)$$

Second fundamental form S

$$S(X, Y) \;\triangleq\; \widehat{\nabla}_X Y - \nabla_X Y \;=\; P^{\perp}_{TM}\left(\widehat{\nabla}_X Y\right)$$

Scalar second fundamental form S_ν

$$S_\nu(X, Y) \;\triangleq\; (S(X, Y), \nu)$$

(For ν a unit normal vector field on M)

LKC's: The general case

$$\mathcal{L}_i(M) = \sum_{j=i}^{N} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} C_{Nnim} \int_{\partial_j M} \int_{S(T_t \partial_j M^{\perp})} \mathrm{Tr}^{T_t \partial_j M} \left(\widehat{R}^m \widehat{S}_{\nu_{N-j}}^{j-i-2m} \right)$$

$$\times 1_{N_t M}(-\nu_{N-j}) \, \mathcal{H}_{N-j-1}(d\nu_{N-j}) \mathcal{H}_j(dt)$$

LKC's: The general case

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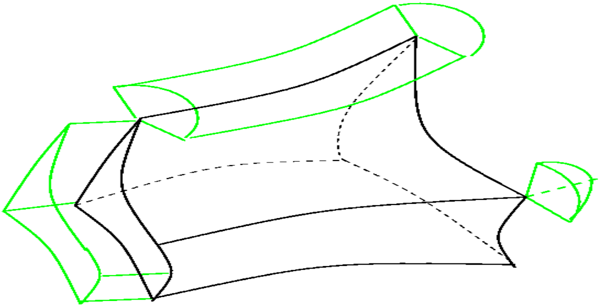
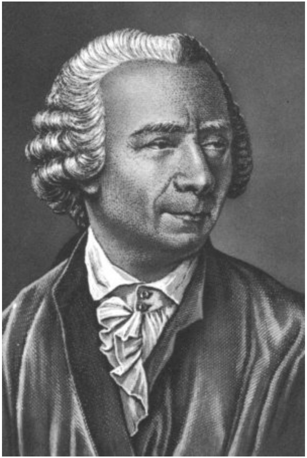




Leonhard Euler
Switzerland
1707-1783

Jules Henri Poincaré
France
1854-1912

$\mathcal{L}_0$: The Euler-Poincaré characteristic

$M \subset \mathbb{R}^N$ is nice, of dimension k, and "triangulisable"

α_0 = number of vertices

α_1 = number of lines

....

....

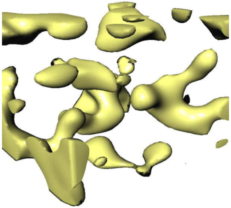
α_k = number of "full" simplices in the triangulation

$\mathcal{L}_0(M) \equiv$ Euler characteristic of M is

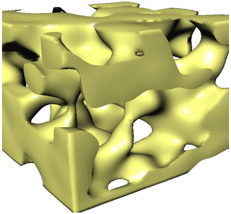
$$\varphi(M) \;=\; \alpha_0 - \alpha_1 + \cdots + (-1)^d \alpha_N$$

3-d excursion sets

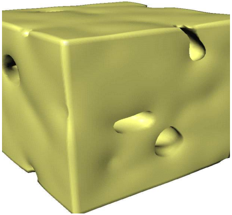
Meatball, EC=21



Sponge, EC=-15



Bubble, EC=1

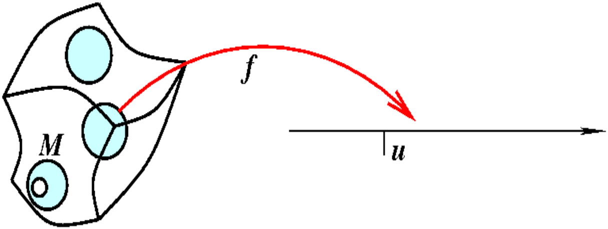


Averaged geometry

of

excursion sets

Excursion sets



$$A_u \;\equiv\; A_u(f, M) \;\stackrel{\Delta}{=}\; \{t \in M:\; f(t) \geq u\}$$

A 30-year old formula

Suppose f is Gaussian, mean zero, variance σ^2, stationary, and isotropic, with second spectral moment λ_2 and $M = [0, T]^N$
Then:

$$\mathbb{E}\left\{\mathcal{L}_0\left(A_u\right)\right\} \;=\; e^{-u^2/2\sigma^2} \sum_{k=1}^{N} \frac{\binom{N}{k} T^k \lambda_2^{k/2}}{(2\pi)^{(k+1)/2}\sigma^k} H_{k-1}\left(\frac{u}{\sigma}\right) + \Psi\left(\frac{u}{\sigma}\right).$$

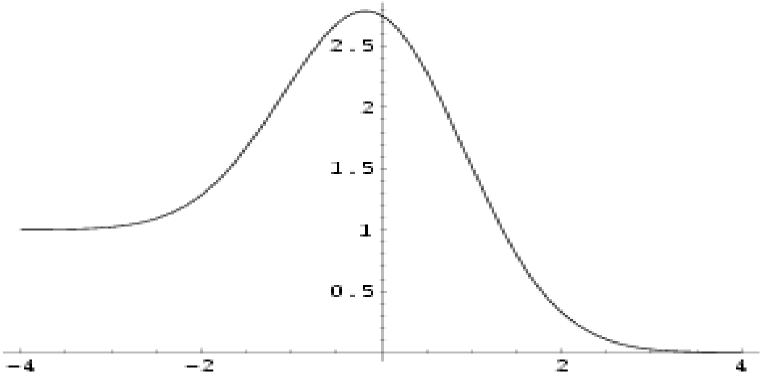
where

$$H_n(x) \;=\; n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j!\,(n-2j)!\,2^j}, \qquad n \geq 0,\; x \in \mathbb{R}$$

$$\Psi(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2}\,dx$$

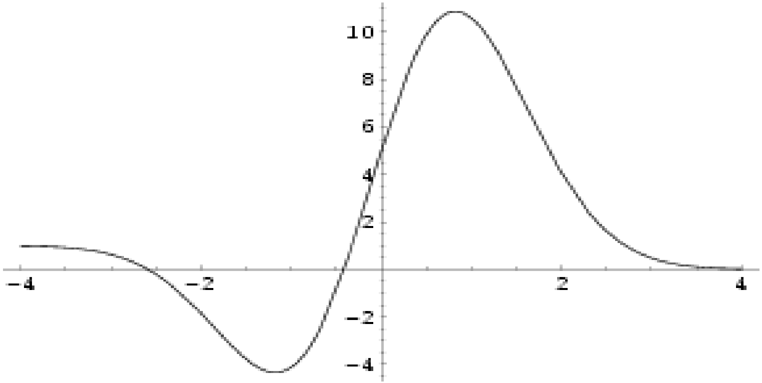
One dimension: A line of length T

$$\mathbb{E}\left\{\mathcal{L}_0\left(A_u(f,[0,T])\right)\right\} \;=\; \Psi(u/\sigma) \;+\; \frac{T\lambda_2^{1/2}}{2\pi\sigma} e^{-u^2/2\sigma^2},$$



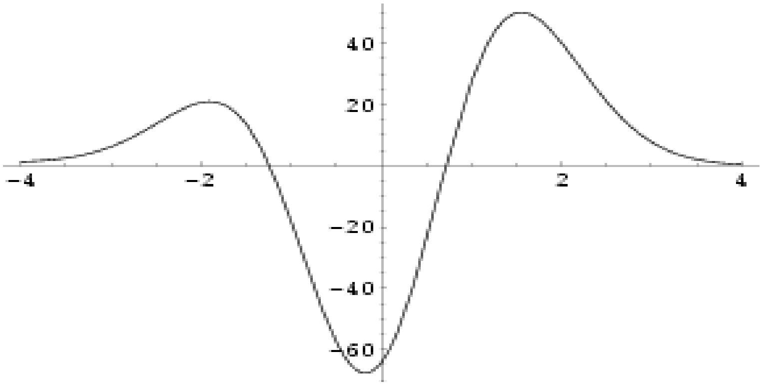
Two dimensions: A square of side length T

$$\left[\frac{T^2\lambda_2}{(2\pi)^{3/2}}\,u \;+\; \frac{2T\lambda_2^{1/2}}{2\pi}\right] e^{-u^2/2} \;+\; \Psi(u).$$



Three dimensions: A cube of side length T

$$\left[\frac{T^3\lambda_2^{3/2}}{(2\pi)^2}\,u^2 \;+\; \frac{3T^2\lambda_2}{(2\pi)^{3/2}}\,u \;+\; \frac{3T\lambda_2^{1/2}}{2\pi} \;-\; \frac{T^3\lambda_2^{3/2}}{(2\pi)^2}\right] e^{-u^2/2} \;+\; \Psi(u).$$



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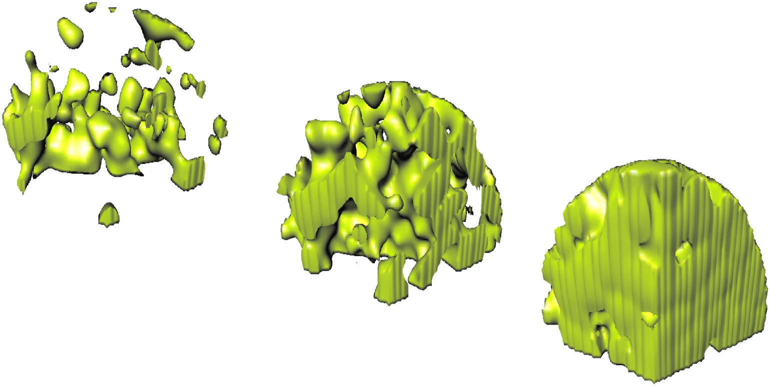
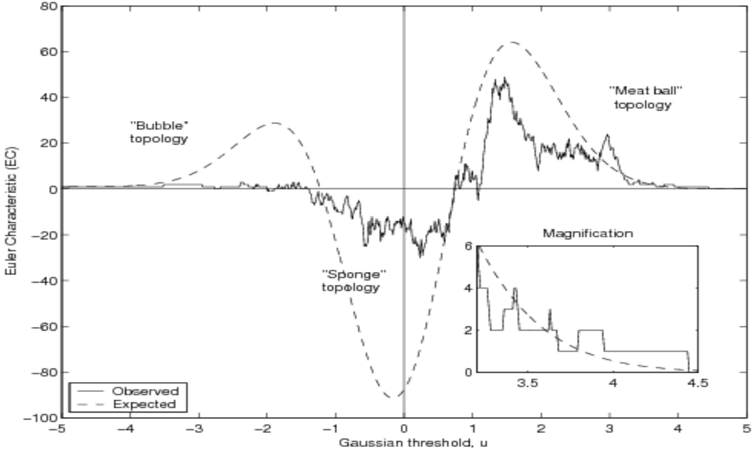
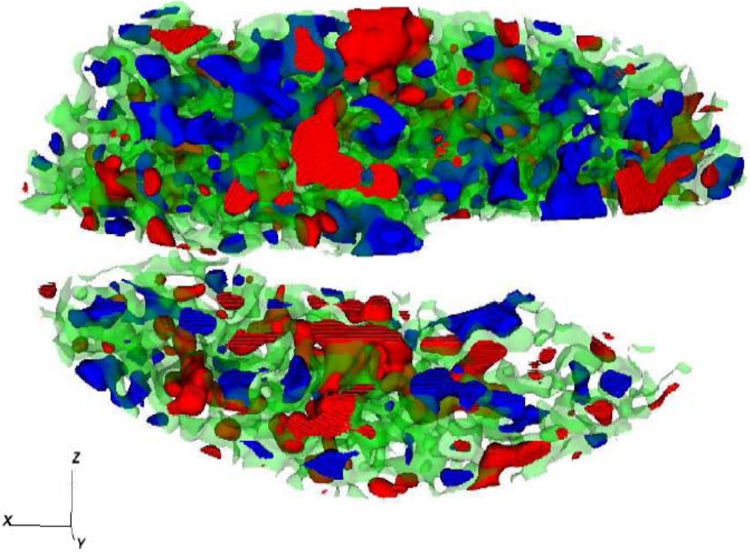
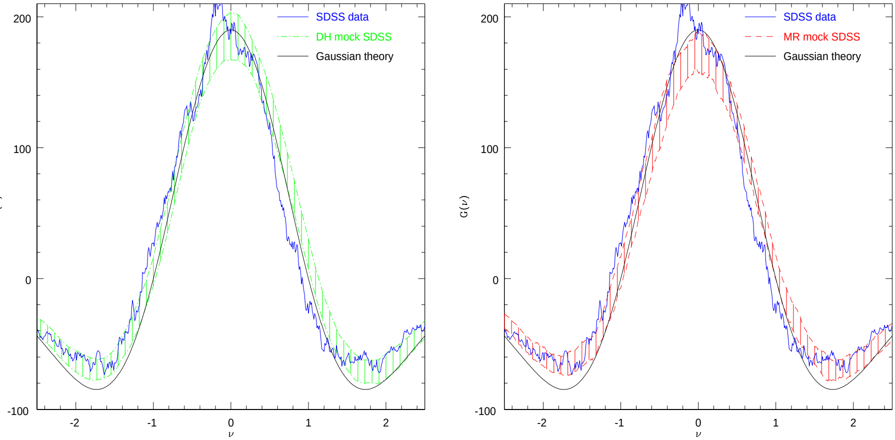




The observed EC of the set of high-density regions of the CfA Galaxy survey. Also shown is the expected EC for randomly distributed galaxies with no structure; the CfA data has smaller EC than expected, indicating less "blobs" and more clumping of galaxies into clusters, strings, and "walls".


Z
X
Y

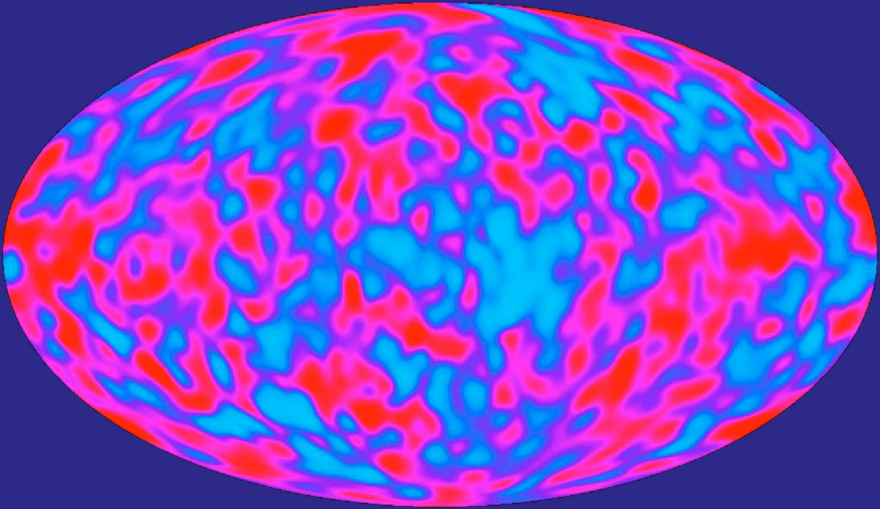


(a) DH

(b) MR

FIG. 7.— Genus curves with shaded 1σ error regions for the (a) 100 DH and (b) 50 MR samples, compared with SDSS and Gaussian
ndom phase.

DMR's Two Year CMB Anisotropy Result



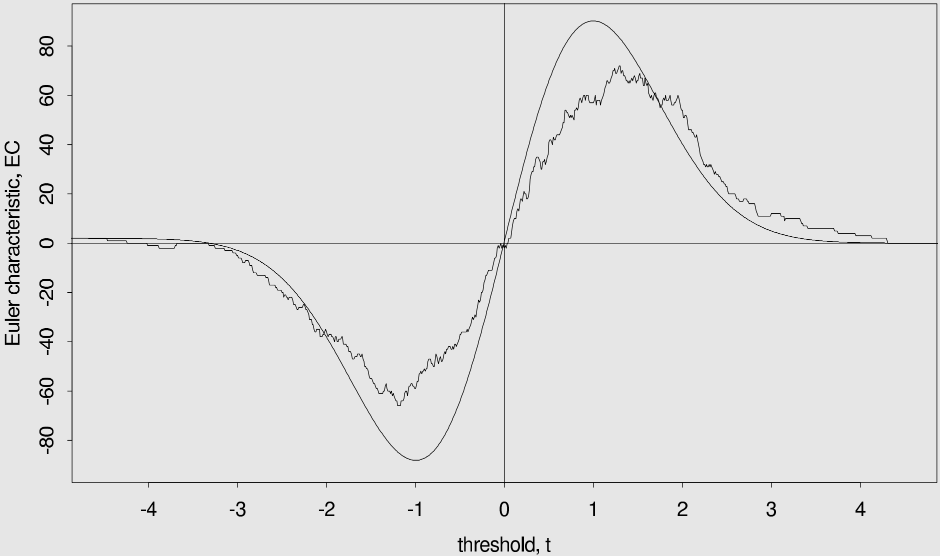
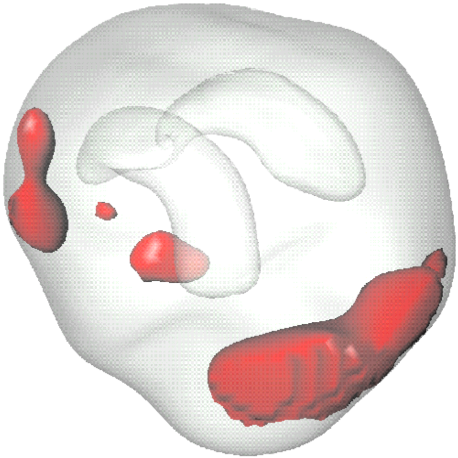


Figure 12. Plot of the observed EC of excursion sets of the anomalies in the cosmic microwave background radiation (jagged line), and the expected EC from the formula (smooth line) if there are no real anomalies. The observed microwave background radiation produces an EC curve similar in shape to that expected, but somewhat lower and spread more in the tails—evidence that some of the anomalies are real and not just due to random noise. This discrepancy points to a Gaussian random field model for the anomalies, with a larger standard deviation and a larger smoothness than the background noise.

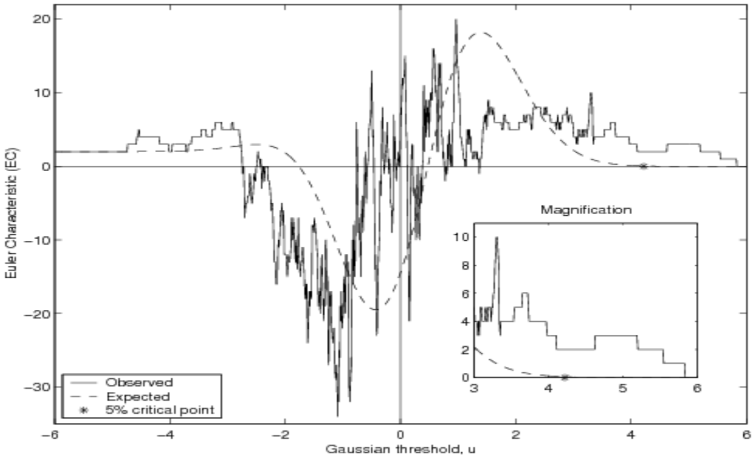


Excursion probabilities

$$\mathbb{P}\left\{\sup_{t\in M} f(t) \ \geq \ u\right\}$$

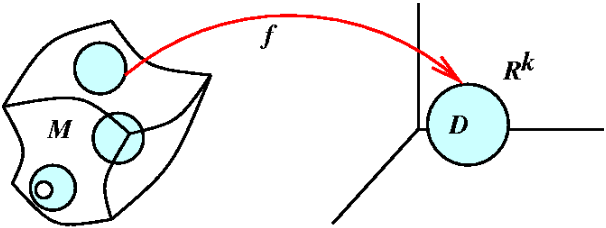
$$\sim \ \mathbb{E}\left\{\mathcal{L}_0\left(A_u(f,M)\right)\right\}$$

$$\liminf_{u\to\infty} u^{-2}\log|\mathbb{P}-\mathbb{E}| \ \geq \ \frac{1}{2}+\frac{1}{2\sigma^2(f)}$$



Observed and expected EC for the PET data and the expected EC if there is no activation due to the linguistic task. In particular, at $u = 3.3$ we expect an EC of 1, but we observe 4. At the 5% critical value of $u = 4.22$, we expect 0.05 but we observe 2 components.

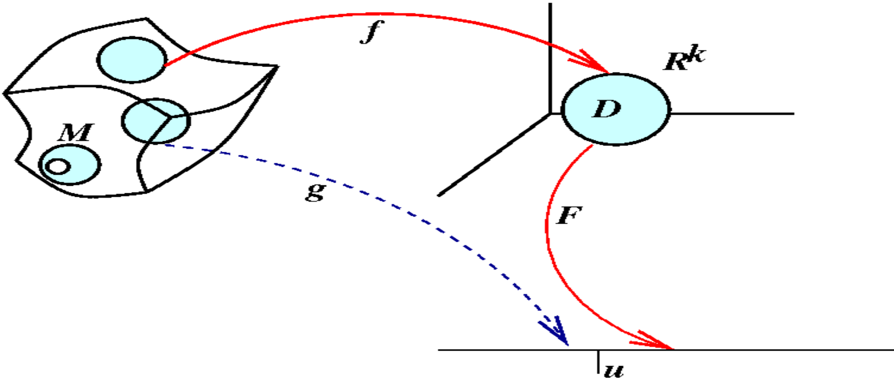
A more general result



$$A_D \equiv A_D(f, M) \stackrel{\Delta}{=} \{t \in M : f(t) \in D\}$$

$$\mathbb{E}\{\mathcal{L}_j(A_D)\} = \sum_{l=0}^{N-j} \begin{bmatrix} j+l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M)\, \mathcal{M}_l^{(k)}(D)$$

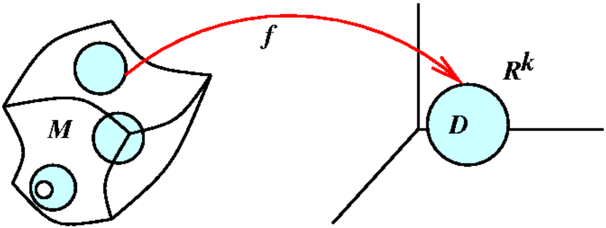
Non-Gaussian fields



$$\sum_{1}^{k} x_i^2, \qquad \frac{x_1\sqrt{k-1}}{(\sum_2^k x_i^2)^{1/2}}, \qquad \frac{m\sum_1^n x_i^2}{n\sum_{n+1}^{n+m} x_i^2}.$$

Now for the mathematics

A more general result



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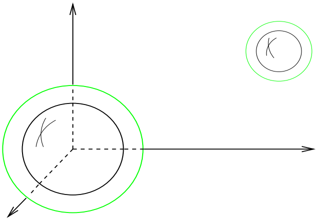
Gaussian tube formula

Gauss measure on $\mathbb{R}^k$

$$\gamma_k(D) \triangleq \frac{1}{(2\pi)^{k/2}} \int_D e^{-\|x\|^2/2}\, dx$$

Gaussian tube formula



$$\gamma_k(\mathrm{Tube}(D,\rho)) = \gamma_k(D) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma}(D)$$

$$\mathbb{E}\left\{\mathcal{L}_j(A_D)\right\} = \sum_{l=0}^{N-j} \begin{bmatrix} j+l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M)\, \mathcal{M}_l^{(k)}(D)$$

Turn M into a Riemannian manifold with

$$g_t(X_t, Y_t) = \mathbb{E}\{X_t f_t \cdot Y_t f_t\}$$

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Curvature tensor:,

$$R(X,Y,Z,W) \;=\; \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\; W\right)$$

Second fundamental form S

$$S(X,Y) \;\stackrel{\Delta}{=}\; \widehat{\nabla}_X Y - \nabla_X Y \;=\; P^{\perp}_{TM}\left(\widehat{\nabla}_X Y\right)$$

Scalar second fundamental form S_ν

$$S_\nu(X,Y) \;\stackrel{\Delta}{=}\; \left(S(X,Y), \nu\right)$$

(For ν a unit normal vector field on M)

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$$\times 1_{N_t M}(-\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j}) \mathcal{H}_j(dt)$$

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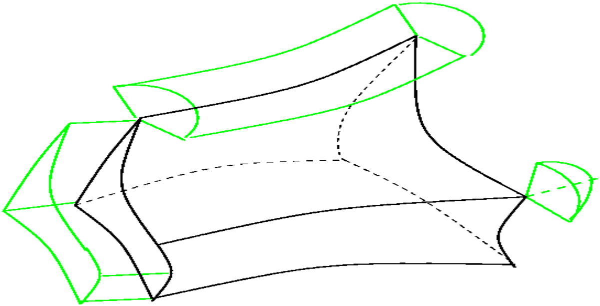
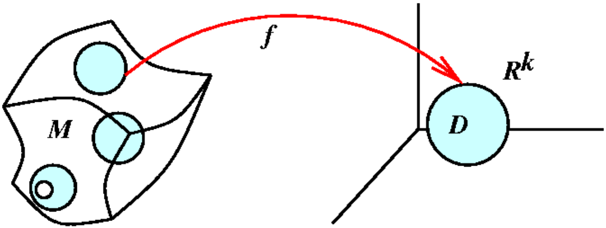


A more general result



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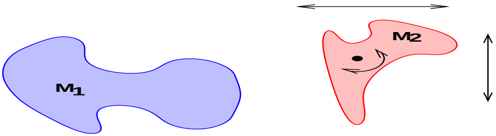
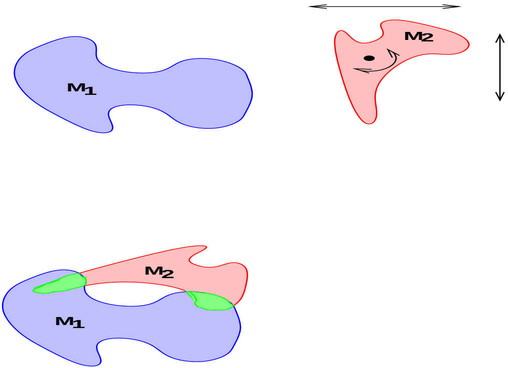
$$\mathbb{E}\{\mathcal{L}_j(A_D)\} = \sum_{l=0}^{N-j} \begin{bmatrix} j+l \\ l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M)\, \mathcal{M}_l^{(k)}(D)$$

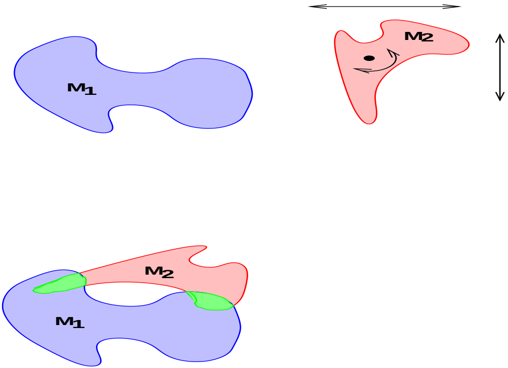
Kinematic fundamental formula



Kinematic fundamental formula

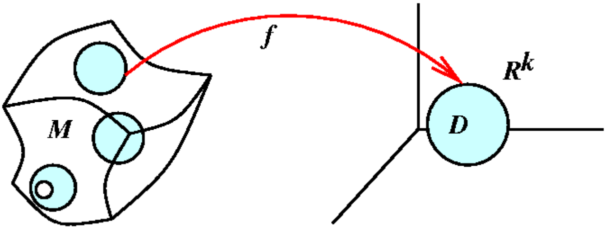


Kinematic fundamental formula



$$\int \mathcal{L}_i\left(M_1 \cap gM_2\right)\, d\nu(g) = \sum_{j=0}^{N-i} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} N \\ j \end{bmatrix}^{-1} \mathcal{L}_{i+j}(M_1)\mathcal{L}_{N-j}(M_2)$$

A more general result



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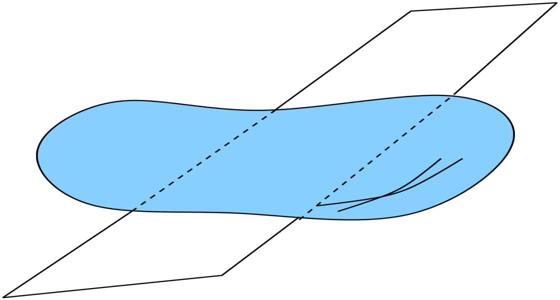
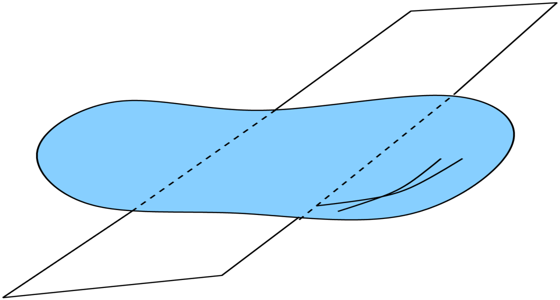
Crofton's formula on $\mathbb{R}^N$

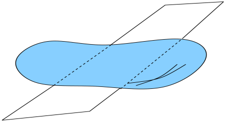


Crofton's formula on $\mathbb{R}^N$



$$\int_{\mathrm{Graff}(N,N-k)} \mathcal{L}_j(M \cap V)\, d\lambda^N_{N-k}(V) \;=\; \begin{bmatrix} k+j \\ j \end{bmatrix} \mathcal{L}_{k+j}(M)$$

A Gauss-Crofton formula



M a C^2, n-dimensional, Riemannian manifold

$y^1, \ldots, y^k$ Gaussian on M, matched to the metric

Define, for $u \in \mathbb{R}^k$, the (random) submanifold

$$D_u = \left\{ t \in M : y_t^1 = u_1, \ldots, y_t^k = u_k \right\}$$

Take $Z_k \in \mathbb{R}^k$ standard Gaussian independent of y

$$D_{Z_k} = \{ t \in M : y_t = Z_k \}$$

$$\mathbb{E}\left\{ \mathcal{L}_j(M \cap D_{Z_k}) \right\} = (2\pi)^{-k/2} \frac{[k+j]!}{[j]!} \mathcal{L}_{k+j}(M)$$

About the proofs

Excursion probabilities

$$\mathbb{P}\left\{\sup_{t\in M} f(t) \;\geq\; u\right\}$$

$$\sim\; \mathbb{E}\left\{\mathcal{L}_0\left(A_u(f,M)\right)\right\}$$

$$\liminf_{u\to\infty} u^{-2}\log|\mathbb{P}-\mathbb{E}| \;\geq\; \frac{1}{2}+\frac{1}{2\sigma^2(f)}$$

Mapping to the sphere

$$f_t \;=\; \sum_{n=1}^{K} \xi_n \varphi_n(t) \;=\; \langle \xi, \varphi(t) \rangle$$

Mapping to the sphere

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Map M into the sphere with

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Map M into the sphere with

$$t \;\rightarrow\; \varphi(t)$$

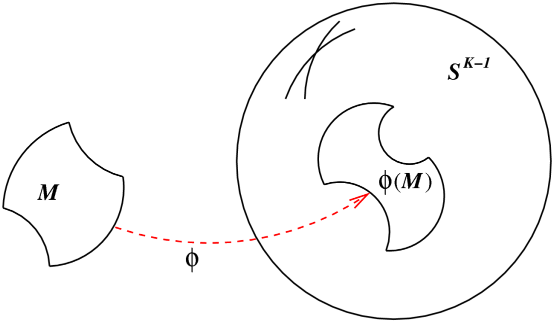
Then

$$g(x) \;\stackrel{\Delta}{=}\; f\left(\varphi^{-1}(x)\right)$$

has covariance

$$\mathbb{E}\left\{g(x)g(y)\right\} = \langle x, y \rangle$$

One (Gaussian) case covers all !?!



$$t \quad \longrightarrow \quad \varphi(t) \ \stackrel{\Delta}{=} \ (\varphi_1(t), \ldots, \varphi_K(t))$$

Laboratoire de
MATHEMATIQUES
UMR 6620 UBP-CNRS

Menu

39th Probability Summer School
Saint-Flour (France), July 5-18, 2009

Photographe : J.F. FERRATON - Saint-Flour

Founded in 1971, this school is organised every year by the **Laboratoire de Mathématiques** (UMR 6620). It is supported by **Blaise Pascal University** (Clermont-Ferrand II), the **Ministry of Research** and the **C.N.R.S.** It is intended for PhD students, teachers and researchers who are interested in probability theory, statistics, and in applications of these techniques.

The school has three main goals:

1. to provide, in three high level courses, a comprehensive study of a field in probability theory or statistics;
2. to enable the participants to explain their work in lectures;

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Webmaster

The lecturers are chosen by the **Scientific Board** of the school.

Lectures 2009

Robert ADLER: Topological complexity of smooth random functions.
Mireille BOUSQUET-MÉLOU: Enumerative combinatorics for probability.
Alison ETHERIDGE: Some mathematical models from population genetics.

Abstracts

Practical information and registration rate

The participants will be lodged at the Maison des Planchettes, 7 rue des Planchettes, 15100 Saint-Flour (France). The lectures will be given at the same place. Full board accommodation is included in the registration fee for the participants; the families can also be lodged, and their accommodation should be paid for during the school.

More information, in particular the registration fee, will be available in February.

Registration

This monograph is devoted to a completely new approach to geometric problems arising in the study of random fields. The groundbreaking material in Part III, for which the background is carefully prepared in Parts I and II, is of both theoretical and practical importance, and striking in the way in which problems arising in geometry and probability are beautifully intertwined.

The three parts to the monograph are quite distinct. Part I presents a user-friendly yet comprehensive background to the general theory of Gaussian random fields, treating classical topics such as continuity and boundedness, entropy and majorizing measures, Borell and Slepian inequalities. Part II gives a quick review of geometry, both integral and Riemannian, to provide the reader with the material needed for Part III, and to give some new results and new proofs of known results along the way. Topics such as Crofton formulae, curvature measures for stratified manifolds, critical point theory, and tube formulae are covered. In fact, this is the only concise, self-contained treatment of all of the above topics, which are necessary for the study of random fields. The new approach in Part III is devoted to the geometry of excursion sets of random fields and the related Euler characteristic approach to extremal probabilities.

Random Fields and Geometry will be useful for probabilists and statisticians, and for theoretical and applied mathematicians who wish to learn about new relationships between geometry and probability. It will be helpful for graduate students in a classroom setting, or for self-study. Finally, this text will serve as a basic reference for all those interested in the companion volume of the applications of the theory. These applications, to appear in a forthcoming volume, will cover areas as widespread as brain imaging, physical oceanography, and astrophysics.

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Random Fields and Geometry

ROBERT J. ADLER
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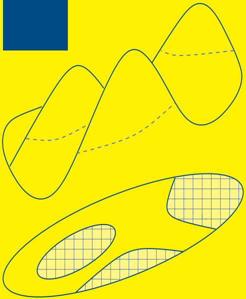
Random Fields and Geometry



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