Anisotropic Random Fields and Their Fractal Dimensions

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1. Fractals generated by random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \qquad t \in \mathbb{R}^N.$$

Then the following random sets are often fractals:

Range $\quad X\left([0,1]^N\right) = \left\{X(t) : t \in [0,1]^N\right\}.$

Graph $\quad \mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)) : t \in [0,1]^N\right\}.$

Level set $\quad X^{-1}(x) = \left\{t \in \mathbb{R}^N : X(t) = x\right\}.$

Excursion set $\quad X^{-1}(F) = \left\{t \in \mathbb{R}^N : X(t) \in F\right\}, \ \forall F \subset \mathbb{R}^d.$

2. Anisotropy and operator-self-similarity

2.1 Operator-self-similarity in space

An (N, d)-random field $\{X(t), t \in \mathbb{R}^N\}$ is called *operator self-similar in the space-variable* if there exists a $d \times d$ matrix $D = (d_{ij})$ such that for all constants $c > 0$,

$$\left\{X(c\,t), t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c^D\, X(t), t \in \mathbb{R}^N\right\}.$$

In the above, c^D is the linear operator defined by

$$c^D = \sum_{n=0}^{\infty} \frac{(\ln c)^n D^n}{n!}.$$

The linear operator D is called a space-variable self-similarity exponent [which may not be unique].

Example: Gaussian fields with fBm components

Let $X_1, \ldots, X_d$ be independent N-parameter fractional Brownian motions with indices $\alpha_1, \ldots, \alpha_d$, respectively.

We define an (N, d)-Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ by

$$X(t) = (X_1(t), \ldots, X_d(t)), \qquad t \in \mathbb{R}^N.$$

Then X is operator-self-similar with D= the diagonal matrix with entries $\alpha_1, \ldots, \alpha_d$ on the diagonal.

When $\alpha_1, \ldots, \alpha_d$ are not the same, X is anisotropic in the space variable.

General operator-fractional Brownian motions were constructed in Mason and Xiao (2001), Bahadoran, Benassi and Dębicki (2003), Didier and Pipiras (2007a, b).

2.2 Operator-self-similarity in time

An (N, d)-random field $\{X(t), t \in \mathbb{R}^N\}$ is called *operator self-similar in the time-variable* if there exists an $N \times N$ matrix E such that for all constants $c > 0$,

$$\left\{X(c^E t), t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c\, X(t), t \in \mathbb{R}^N\right\}.$$

The linear operator E is called a time-variable self-similarity exponent [which may not be unique].

Examples:

fractional Brownian sheets;

solution to stochastic heat equation;

Biermé, Meerschaert and Scheffler (2007).

Fractional Brownian sheets

$W^H = \left\{W^H(t), t \in \mathbb{R}^N\right\}$ is a mean 0 Gaussian random field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^H(s)W^H(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right),$$

where $H = (H_1, \ldots, H_N) \in (0,1)^N$ is called the Hurst index.

W^H has the *operator-scaling property*: For all constants $c > 0$,

$$\left\{W^H(c^A t),\, t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c^N W^H(t),\ t \in \mathbb{R}^N\right\},$$

where $A = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/H_i$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \ne j$.

3. Dependence structures of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \qquad t \in \mathbb{R}^N, \tag{1}$$

where $X_1, \ldots, X_d$ are independent copies of X_0.

Denote

$$\sigma^2(s,t) = \mathbb{E}(X_0(s) - X_0(t))^2.$$

Given constants $0 < H_1 \le \ldots \le H_N < 1$, define a metric ρ on $\mathbb{R}^N$:

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in \mathbb{R}^N. \tag{2}$$

Three basic conditions

(C1). $\exists$ positive constants c_1 and c_2 such that for all $s, t \in I := [\varepsilon, 1]^N$,

$$c_1 \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \le \sigma^2(s,t) \le c_2 \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \qquad (3)$$

(C2). $\exists$ $c_3 > 0$ such that for all $s, t \in [\varepsilon, 1]^N$,

$$\mathrm{Var}(X_0(t)|X_0(s)) \ge c_3 \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \qquad (4)$$

(C3). $\exists\ c_4 > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}(X_0(u) \,|\, X_0(t^1), \ldots, X_0(t^n)) \geq c_4 \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left| u_j - t_j^k \right|^{2H_j},$$

where $t_j^0 = 0$ for every $j = 1, \ldots, N$.

or

(C3′). $\exists\ c_5 > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}(X_0(u) \,|\, X_0(t^1), \ldots, X_0(t^n)) \geq c_5 \min_{0 \leq k \leq n} \rho(u, t^k)^2,$$

where $t^0 = 0$ and ρ is the metric on $\mathbb{R}^N$ defined in (2).

4. Dimensions of the range, graph and level sets

Theorem 4.1 [Ayache and Xiao (2005), Xiao (2007)] Let X be defined by (1) such that X_0 satisfies Condition (C1). Then almost surely

$$\dim_{\mathrm{H}} X\left([0,1]^N\right) = \min\left\{d; \quad \sum_{j=1}^{N}\frac{1}{H_j}\right\}, \tag{5}$$

and

$$\begin{aligned}&\dim_{\mathrm{H}} \mathrm{Gr} X\left([0,1]^N\right)\\ &= \min_{1\le k\le N}\left\{\sum_{j=1}^{k}\frac{H_k}{H_j} + N - k + (1-H_k)d; \ \sum_{j=1}^{N}\frac{1}{H_j}\right\},\end{aligned}$$

where $\sum_{j=1}^{0}\frac{1}{H_j} := 0$.

Remarks

• The above results are significantly different from the isotropic case, as well as the space-anisotropic case.

• If $d = 1$, then

$$\dim_{\mathrm{H}} \mathrm{Gr} X\left([0,1]^N\right) = N + 1 - H_1 \qquad \text{a.s.}$$

Hence one needs $d > 1$ to recover all the parameters $H_1, \ldots, H_N$.

• To determine $\dim_{\mathrm{H}} X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}^N$, one needs to use a different Hausdorff-type dimension; see Wu and Xiao (2007).

Theorem 4.2 [Ayache and Xiao (2005), Xiao (2007)] If X_0 satisfies Conditions (C1) and (C2), then

(i). If $\sum_{j=1}^{N} \frac{1}{H_j} < d$, then for every $x \in \mathbb{R}^d$, $X^{-1}(x) = \emptyset$ a.s.

(ii). If $\sum_{j=1}^{N} \frac{1}{H_j} > d$, then for any $x \in \mathbb{R}^d$ and $0 < \varepsilon < 1$, with positive probability

$$\begin{aligned} &\dim_{\mathrm{H}} \left(X^{-1}(x) \cap [\varepsilon, 1]^N \right) \\ &= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \quad \text{if} \quad \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j}. \end{aligned}$$

If $d = 1$, then $\dim_{\mathrm{H}} \left(X^{-1}(x) \cap [\varepsilon, 1]^N \right) = N - H_1$.

5. Hitting probability

The following result is proved in Biermé, Lacaux and Xiao (2007) [see also Xiao (2007)].

Theorem 5.1 If X is the Gaussian random field defined by (1) such that X_0 satisfies Conditions (C1) and (C2) on $I = [\varepsilon, 1]^N$. Then for any Borel set $F \subset \mathbb{R}^d$,

$$c_6\, \mathcal{C}_{d-Q}(F) \le \mathbb{P}\{X(I) \cap F \ne \emptyset\} \le c_7\, \mathcal{H}_{d-Q}(F), \qquad (6)$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, $\mathcal{C}_{d-Q}$ is Riesz capacity and $\mathcal{H}_{d-Q}$ denotes $(d-Q)$-dimensional Hausdorff measure, respectively.

Results of this type are interesting in the potential theory for random fields.

The following is a corollary of Theorem 5.1.

Corollary 5.2 Assume the conditions of Theorem 5.1 hold. Then for any Borel set $F \subset \mathbb{R}^d$,

$$\mathbb{P}\Big\{X(I) \cap F \neq \emptyset\Big\} \begin{cases} = 0 & \text{if } \dim_{\mathrm{H}} F < d - Q, \\ > 0 & \text{if } \dim_{\mathrm{H}} F > d - Q. \end{cases}$$

6. Hausdorff dimension of the inverse images

Theorem 6.1 [Biermé, Lacaux and Xiao (2007)] Suppose Conditions (C1) and (C2) hold. Let $F \subseteq \mathbb{R}^d$ be a Borel set such that $\sum_{j=1}^{N} \frac{1}{H_j} > d - \dim_{\rm H} F$. Then with positive probability,

$$
\begin{aligned}
&\dim_{\rm H}\Big(X^{-1}(F) \cap I\Big) \\
&= \min_{1\le k\le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F) \right\} \\
&= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F), \\
&\qquad\qquad \text{if} \quad \sum_{j=1}^{k-1} \frac{1}{H_j} \le d - \dim_{\rm H} F < \sum_{j=1}^{k} \frac{1}{H_j}.
\end{aligned}
$$

7. Proofs of Theorems 5.1 and 6.1

For proving the upper bounds in Theorems 5.1 and 6.1, we make use of the following lemma together with a covering argument.

Lemma 7.1 Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying Conditions (C1) and (C2). Then there exists a constant $c_8 > 0$ such that for all $t \in I$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}\left\{ \inf_{s \in B_\rho(t,r)} \|X(s) - x\| \le r \right\} \le c_8\, r^d.$$

In the above $B_\rho(t, r) = \{s \in \mathbb{R}^N : \rho(s, t) \le r\}$ denotes the ball of radius r in the metric ρ in $\mathbb{R}^N$ defined by (2).

For proving the lower bounds in Theorems 5.1 and 6.1, we construct a random Borel measure on $X^{-1}(F) \cap I$, and show this measure is positive with positive probability.

Let ν be a Borel measure on F. For all integers $n \geq 1$, the random measure μ_n on I is defined by

$$\begin{aligned}
&\int_I f(t)\,\mu_n(dt) \\
&= \int_I \int_{\mathbb{R}^d} (2\pi n)^{\frac{d}{2}} \exp\big(-n\,\|X(t)-x\|^2\big)\, f(t)\,\nu(dx)\,dt \\
&= \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{\|\xi\|^2}{2n} + i\langle \xi, X(t)-x\rangle\right) f(t)\,d\xi\,\nu(dx)\,dt,
\end{aligned}$$

where f is an arbitrary measurable, nonnegative function on I.

The following lemma is useful.

Lemma 7.2 Under the conditions of Lemma 7.1, there exists a constant $c_9 > 0$ such that for all $\varepsilon \in (0, 1)$, $s, t \in I$ and $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^{2d}} e^{-i(\langle \xi, x\rangle + \langle \eta, y\rangle)} \exp\left(-\frac{1}{2}(\xi,\, \eta)\, (\varepsilon I_{2d} + \mathrm{Cov}(X(s), X(t)))\, (\xi,\, \eta)'\right) d\xi d\eta \le \frac{c_9}{\max\{\rho^d(s,t),\, \|x-y\|^d\}},$$

where I_{2d} is the identity matrix of order $2d$, $\mathrm{Cov}(X(s), X(t))$ is the covariance matrix of $(X(s), X(t))$ and $(\xi, \eta)'$ is the transpose of the vector (ξ, η).

8. Further results

8.1. Stable random fields

Theorem 4.1 still holds for

- linear fractional stable sheets (Ayache, Roueff and Xiao, 2008) with $H_j > 1/\alpha$.

- harmonizable fractional stable sheets (work in progress by Ayache, Shieh and Xiao, 2009).

The conclusions of Theorems 4.2, 5.1 and 6.1 can be proven to be partially true for harmonizable fractional stable sheets.

8.2. Packing dimension results

Packing dimension was introduced by Tricot (1982), Taylor and Tricot (1985).

It is interesting to determine the packing dimensions of the random sets, which may be very different from the Hausdorff dimension results.

There are only a few results for Gaussian random fields.

- Isotropic Gaussian fields (Talagrand and Xiao, 1996; Xiao, 1997; 2008).

- Anisotropic Gaussian fields (work in progress by Estrade, Wu and Xiao, 2009).

Thank You!